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Infinite ergodic theory

Examples: One-dimensional maps with indifferent fixed points

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Summary: The purpose of these lectures is to illustrate several aspects of infinite ergodic theory by means of one-dimensional maps with indifferent fixed points, and to present some concepts and results from the general theory which are particularly relevant to these maps. The material about the examples is taken from the author's papers. For topics from general infinite ergodic theory the standard reference is

J. Aaronson, *An Introduction to Infinite Ergodic Theory.* Mathematical Surveys and Monographs 50, Amer. Math. Soc., 1997

At the end of each part there is a list of sources used to prepare these notes. To get a more complete picture the reader is asked to form the union over all reference lists contained in these items.

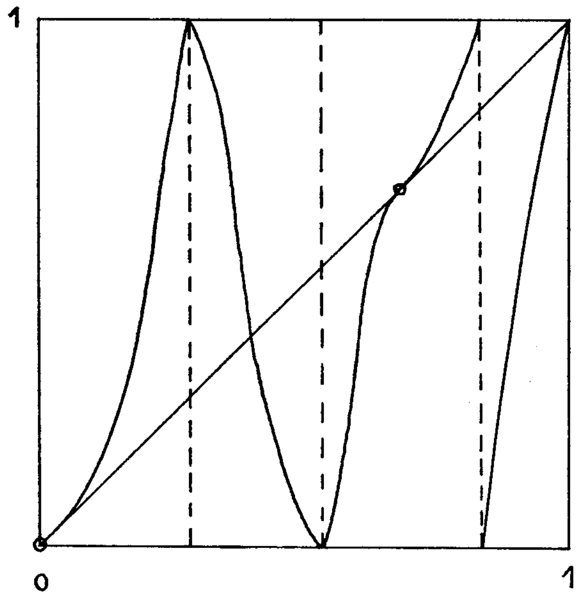
The author would like to thank Mrs Eva-Maria Dannbauer for typing these notes within a record time.

Salzburg, January 2001

Contents

The class of maps and some examples

We shall consider maps $T: [0,1] \to [0,1]$ of the following type:



There exists a finite or infinite family $\xi_1 = \{Z_k : k \in I\}$ of pairwise disjoint subintervals of $[0,1]$ such that

$$\lambda\left(\bigcup_{k\in I} Z_k\right) = 1$$

($\lambda \ldots$ Lebesgue measure).

T is assumed to satisfy the following conditions:

(T1) $T_{|Z_k}$ is twice differentiable, and $\overline{TZ_k} = [0,1]$ for all $k \in I$.

(T2) There exists a non-empty finite set $J \subseteq I$ such that $Z_j, j \in J$, contains a fixed point x_j with $T'(x_j) = 1$ (indifferent fixed point).

(T3) $|T'| \geq \rho(\varepsilon) > 1$ on $\bigcup_{k\in I} Z_k \setminus \bigcup_{j\in J}(x_j - \varepsilon, x_j + \varepsilon)$, $\forall \varepsilon > 0$.

(T4) $\exists\, \eta > 0$ such that for all $j \in J$,
T' is decreasing on $(x_j - \eta, x_j) \cap Z_j$, and
T' is increasing on $(x_j, x_j + \eta) \cap Z_j$ (x_j is a regular source).

(T5) $T''/(T')^2$ is bounded on $\bigcup_{k\in I} Z_k$ (Adler's condition).

More general classes, in particular, regarding the range structure, are considered in

- J. Aaronson's book, Chapter 4: Markov maps (see additional references there)
- R. Zweimüller: $[Z1], [Z2]$ (non-Markovian case).

The purpose here is to concentrate on the question how the indifferent fixed points influence the ergodic (stochastic) properties of the system.

A map T satisfying (T1) - (T5) is a non-singular transformation on the measure space $([0,1], \mathcal{B}, \lambda)$ where (in this situation) $\mathcal{B}$ denotes the Borel σ-field on $[0,1]$.

We call T a *non-singular transformation* on the measure space $(X, \mathcal{B}, m)$ if there exists a set $X_0 \in \mathcal{B}$ with $m(X \backslash X_0) = 0$ such that $T: X_0 \to X$ is measurable and $m(T^{-1}A) = 0$ for all $A \in \mathcal{B}$ with $m(A) = 0$. T then also denotes any measurable extension to X. The same agreement is made regarding measure preserving transformations.

In the following we mention a few examples which motivated a unifying approach. For all these examples the λ-absolutely continuous invariant measure μ is known explicitely, and enables conservativity (recurrence) to be seen immediately by means of

Maharam's recurrence theorem ([Aa1], p. 19): Let T be a measure preserving transformation on a measure space $(X, \mathcal{B}, \mu)$, and suppose there exists a set $A \in \mathcal{B}$ with $\mu(A) < \infty$ such that $\bigcup_{n=0}^{\infty} T^{-n}A = X \pmod{\mu}$. Then T is conservative.

Proof: We have $\bigcup_{n=N}^{\infty} T^{-n}A = X \pmod{\mu}$ for all $N \geq 0$, and thus $\limsup_n T^{-n}A = X \pmod{\mu}$, i.e.,

$$\sum_{n=0}^{\infty} 1_A \circ T^n = \infty \qquad \text{a.e. on } X.$$

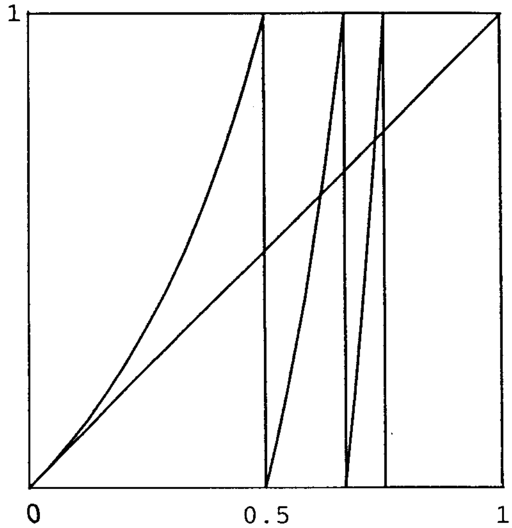
Let $W \in \mathcal{B}$ be a wandering set, i.e., $\sum_{n=0}^{\infty} 1_W \circ T^n \leq 1$. Then, for all $n \geq 0$,

$$\begin{aligned} \infty > \mu(A) = \mu(T^{-n}A) &\geq \int_{T^{-n}A} \sum_{k=0}^{n} 1_W \circ T^k \, d\mu \\ &= \sum_{k=0}^{n} \int \left(1_A \circ T^{n-k} \cdot 1_W\right) \circ T^k \, d\mu \\ &= \int_W \sum_{k=0}^{n} 1_A \circ T^k \, d\mu . \quad \text{Thus, } \mu(W) = 0. \end{aligned}$$

□

Example 1 (A. Rényi, 1957; Hungarian version of [R])

$$T(x) = \frac{x}{1-x} \pmod 1$$



$$Z_k = \left[1 - \frac{1}{k+1}, 1 - \frac{1}{k+2}\right[,$$

$$k = 0, 1, 2, \ldots$$

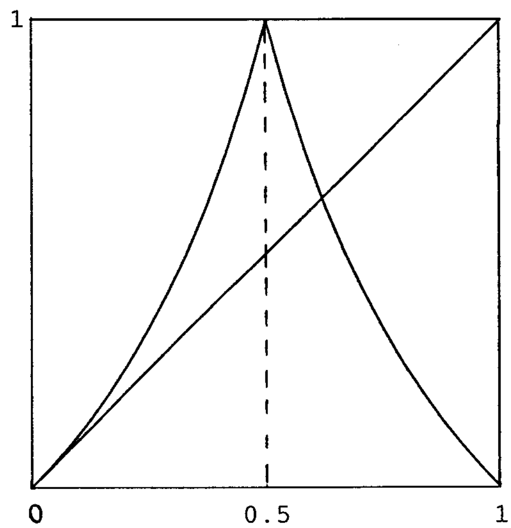
$$J = \{0\}$$

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{x}$$

$(Tx = x + x^2 + \ldots \text{ at } 0)$

Example 2 (H. E. Daniels ([D]), W. Parry ([P2]), 1962)

$$T(x) = \begin{cases} \dfrac{x}{1-x}, & x \in [0, 1/2] \\ \dfrac{1}{x} - 1, & x \in (1/2, 1] \end{cases}$$



$$Z_0 = [0, 1/2], \quad Z_1 = (1/2, 1]$$

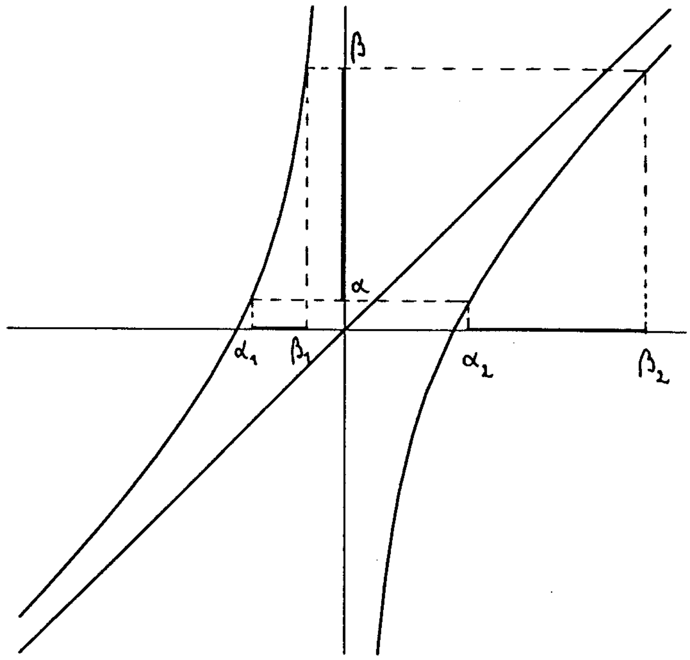
$$J = \{0\}$$

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{x}$$

((T3) is not satisfied literally as $T'(1) = -1$.)

Example 3: Boole's transformation

$T:\ \mathbb{R} \to \mathbb{R}:\quad T(x) = x - \dfrac{1}{x}$



$\dfrac{d\mu}{d\lambda} = 1:$

$T^{-1}([\alpha,\beta]) = [\alpha_1,\beta_1] \cup [\alpha_2,\beta_2];$

α_1, α_2 are the solutions of

$Tx = \alpha$, i.e., $x^2 - \alpha x - 1 = 0$.

By Vieta's theorem,

$\alpha_1 + \alpha_2 = \alpha$ (and $\beta_1 + \beta_2 = \beta$).

Thus,

$\lambda\Big(T^{-1}([\alpha,\beta])\Big) = (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) = \beta - \alpha = \lambda([\alpha,\beta])\quad \square$

Origin: G. Boole, 1857 ([Boo])

$$\int_{-\infty}^{+\infty} f\left(x - \frac{a_1}{x-\lambda_1} - \ldots - \frac{a_n}{x-\lambda_n}\right) dx = \int_{-\infty}^{+\infty} f(x)dx,\ f \text{ integrable}$$

$(a_1,\ldots,a_n > 0,\ \lambda_1,\ldots,\lambda_n \in \mathbb{R},\ n \geq 1)$, i.e., λ is invariant for

$$x \mapsto x - \frac{a_1}{x-\lambda_1} - \ldots - \frac{a_n}{x-\lambda_n},\quad x \in \mathbb{R}.$$

Ergodicity of T: R. L. Adler – B. Weiss, 1973 ([AW]; see also [Ad])
F. Schweiger, 1975 ([Schl])

Deeper analysis: J. Aaronson, 1978 ([Aa2])

T is the $\mathbb{R}$-restriction of an inner function of the upper half plane, and can therefore be studied in a very elegant way using complex analysis. We refer to Chapter 6 of J. Aaronson's book for this approach.

Change of variables

T belongs to our class up to conjugation. E.g.,

$$T = \phi \circ \tilde{T} \circ \phi^{-1} \quad \text{with} \quad \phi(x) = \frac{1}{1-x} - \frac{1}{x}, \quad x \in (0,1),$$

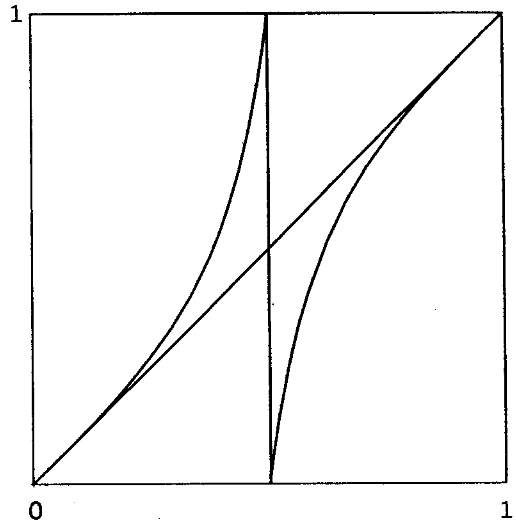
$\tilde{T}(x) = \dfrac{x(1-x)}{1-x-x^2}$, $x \in [0, 1/2]$,

$\tilde{T}(x) = 1 - \tilde{T}(1-x)$, $x \in (1/2, 1]$,

$$\begin{aligned} \frac{d\mu}{d\lambda}(x) &= \frac{1}{x^2} + \frac{1}{(1-x)^2} \\ &= \frac{g(x)}{x^2(1-x)^2}, \end{aligned}$$

g continuous and positive on $[0,1]$

($\tilde{T}x = x + x^3 + \ldots$ at 0)



Example 4: $T(x) = \tan x$, $x \in \mathbb{R}$

J. W. Glaisher, 1870 ([Gl]):

$$\int_{-\infty}^{+\infty} f(\tan x)\frac{dx}{x^2} = \int_{-\infty}^{+\infty} f(x)\frac{dx}{x^2}, \quad \frac{f(x)}{x^2} \text{ integrable},$$

i.e., $\dfrac{1}{x^2}$ is the density of an invariant measure for T.

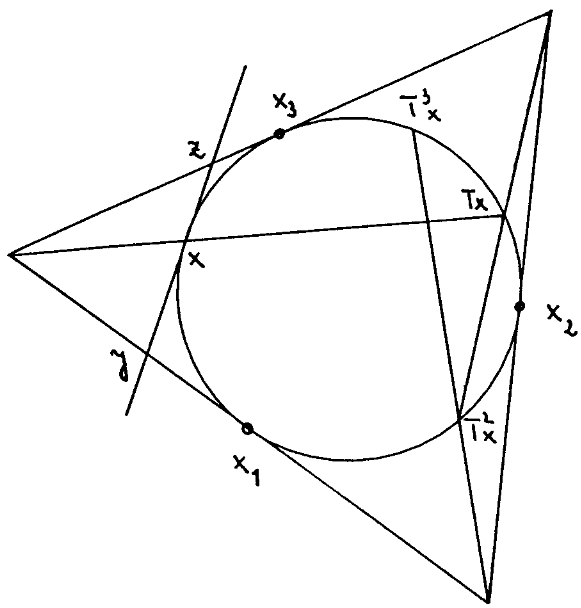
Ergodicity of T: F. Schweiger, 1978 ([Sch3])

Deeper analysis: J. Aaronson, 1978 ([Aa2] and Chapter 6 of [Aa1])

A change of variables (e.g. by $\phi(x) = \tan \pi(x - \frac{1}{2})$, $x \in (0, 1)$) yields a map belonging to our class.

Example 5 (O. I. Bogoyavlensky, 1976)

T acts on a circle in the following way:



x_1, x_2, x_3 are indifferent fixed points (circumscribed triangle not necessarily equilateral); μ infinite

R. Kołodziej, 1981 ([Ko]; see also [Mi]):

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{|xy|} + \frac{1}{|xz|}$$

The proof is based on the Theorem of Ptolemaeus on quadrilaterals inscribed in a circle.

(To treat this example slightly more general conditions are required.)

Example 6 (P. Manneville, 1980 ([Ma]); M. Thaler, 1995 ([Th3]))

$$T(x) = (1+\varepsilon)x + (1-\varepsilon)x^2 \pmod 1, \quad 0 \le \varepsilon \le 1$$

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{\varepsilon + (1-\varepsilon)x} + \frac{1}{1 + (1-\varepsilon)x}$$

$$\varepsilon = 0 : \quad T(x) = x + x^2 \pmod 1, \quad \frac{d\mu}{d\lambda}(x) = \frac{1}{x} + \frac{1}{1+x} = \frac{g(x)}{x},$$

g continuous and positive on $[0, 1]$.

1. Basic ergodic properties

The main purpose of this part is to explain the *method of auxiliary transformations*, which we shall apply to prove the following result.

Theorem ([Th1], [Th2]): Let $T : [0,1] \to [0,1]$ satisfy (T1) - (T5). Then the following assertions hold.

(1) T is conservative and exact with respect to λ.

(2) T has a unique σ-finite invariant measure $\mu \sim \lambda$, and $\mu([0,1]) = \infty$.

(3) The invariant density $\frac{d\mu}{d\lambda}$ has a version h of the form

$$h(x) = h_0(x) \prod_{j \in J} \frac{x - x_j}{x - u_j(x)}, \quad x \in [0,1] \backslash \{x_j : j \in J\},$$

where $u_j = (T_{|Z_j})^{-1}$, $j \in J$, and h_0 is continuous and positive on $[0,1]$.

If $T(x) = x \pm a_j |x - x_j|^{p_j + 1} + o(|x - x_j|^{p_j+1})(x \to x_j)$ with $a_j > 0$, $p_j \geq 1$ $(j \in J)$, then

$$h(x) = g(x) \prod_{j \in J} |x - x_j|^{-p_j}, \quad x \in [0,1] \backslash \{x_j : j \in J\},$$

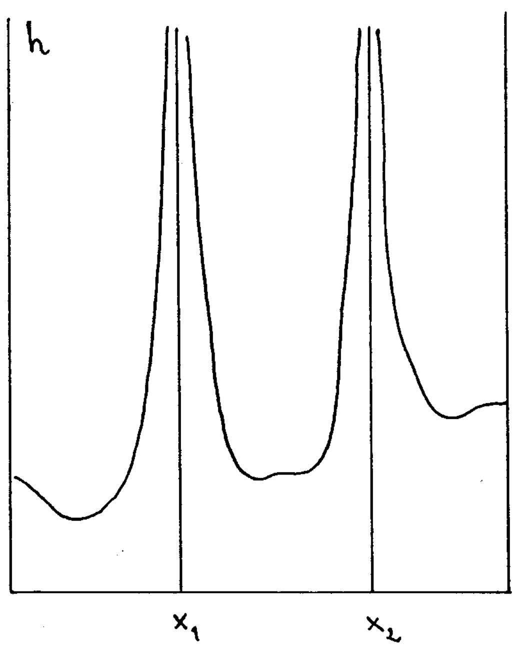
where g is continuous and positive on $[0,1]$. (Compare with the examples.)

Remarks: Basically, the invariant density has a shape as in the picture, with $\int_U h \, d\lambda = \infty$ for each neighbourhood U of x_j, $j \in J$. To see this, note that condition (T5) implies

$|x - u_j(x)| \leq$ const. $(x - x_j)^2$,

$x \in (0,1)$, $j \in J$.

The formula for h can also be derived for other classes of maps, covering examples with indifferent fixed points and finite invariant measure.

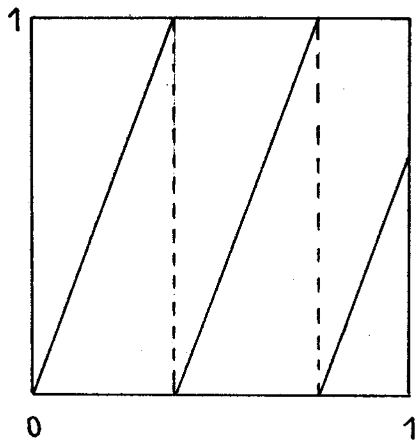


(1.1) Auxiliary transformations

Given a map $T: X \to X$, auxiliary transformations are obtained by replacing T on suitable subsets by suitable iterates of T. Widely known examples are induced transformations and jump transformations. The resulting maps are often simpler from the ergodic theoretic viewpoint, and various properties of the original maps can be deduced from them.

(1.1.1) Motivating examples

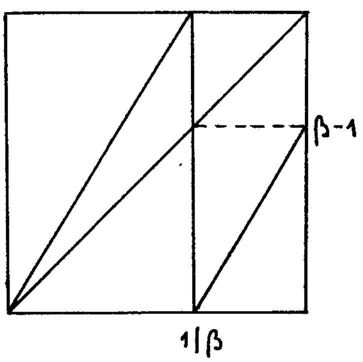
1. <u>β-transformations:</u> $T(x) = \beta x \pmod 1$, $\beta > 1$, $\beta \notin \mathbb{N}$ ([R], [P1],[F])

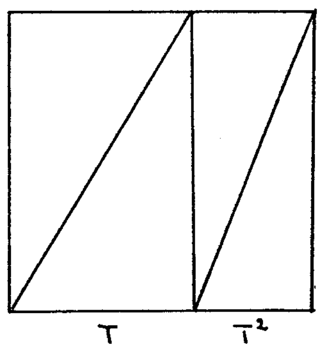


<u>Difficulty:</u> range structure

<u>Remedy:</u> Replacing T on the interval $[[\beta]/\beta, 1[$ by suitable iterates yields a map with full branches.

Simple case: $\beta = \dfrac{1+\sqrt{5}}{2}$ $\quad \left(\dfrac{1}{\beta} = \beta - 1;\ 1 = \dfrac{1}{\beta} + \dfrac{1}{\beta^2}\right)$





$$S(x) = \begin{cases} T(x), & x \in \left[0, \dfrac{1}{\beta}\right[\\ T^2(x), & x \in \left[\dfrac{1}{\beta}, 1\right[\end{cases}$$

Evidently, S is stochastically simpler than T. In particular, λ is invariant for S.

General case:

$$
\begin{aligned}
Z_k &= \Big[\frac{k}{\beta},\frac{k+1}{\beta}\Big[, \quad 0\le k<[\beta], \qquad Z_{[\beta]}=\Big[\frac{[\beta]}{\beta},1\Big]\,;\\
\varepsilon_j &= k :\Leftrightarrow T^{j-1}(1)\in Z_k \quad (j\ge 1),\\
\text{then} \quad & \sum_{j=1}^{\infty}\frac{\varepsilon_j}{\beta^j}=1. \quad \text{Define } S \text{ by}\\
S(x) &= T^n(x),\ x\in\Big[\sum_{j=1}^{n-1}\frac{\varepsilon_j}{\beta^j},\sum_{j=1}^{n}\frac{\varepsilon_j}{\beta^j}\Big[,\ n\ge 1.
\end{aligned}
$$

The process terminates iff $T^n 1 = 0$ for some n (Markov case). In *any* case, however, the resulting map has full branches. Thus, auxiliary transformations may be employed to reduce the study of non-Markovian maps to the Markovian case.

2. Daniels–Parry transformation

$$
\begin{aligned}
T(x) &= \begin{cases} \dfrac{x}{1-x}, & x\in[0,1/2]=Z_0\,,\\[2ex] \dfrac{1}{x}-1, & x\in(1/2,1]=Z_1\,,\end{cases}\\
\frac{d\mu}{d\lambda}(x) &= \frac{1}{x}.
\end{aligned}
$$

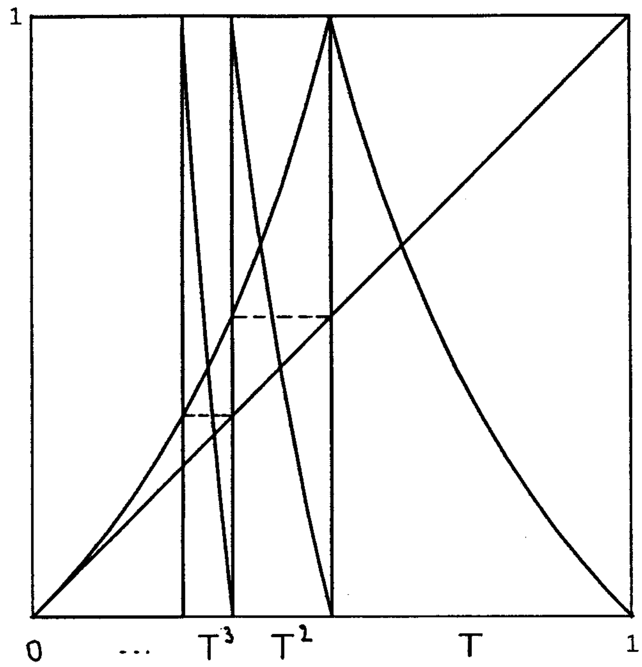
Difficulty: The indifferent fixed point 0 causes long sojourns near 0.

Remedy ([Sch1]): Jumping over runs of visits to Z_0 ('speeding up') neutralizes the effect of the indifferent fixed point, formally:

$$S(x)=T^n x \ \text{ if } \ T^j(x)\in Z_0 \text{ for } 0\le j\le n-2 \text{ and } T^{n-1}(x)\in Z_1$$

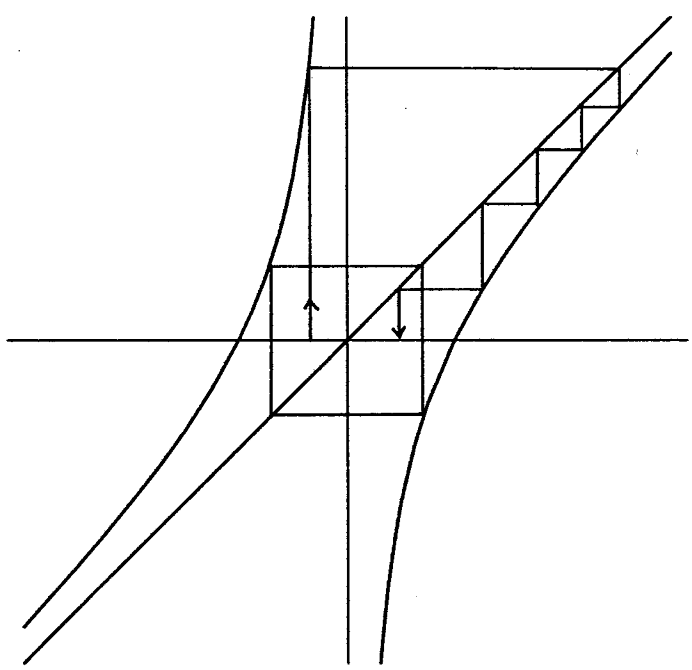
(picture see next page).

It turns out that $S(x) = \frac{1}{x}$ (mod 1), i.e., S is the map associated with ordinary continued fractions (Gauß map). From the well known ergodic properties of S we can easily deduce ergodic properties of T. For example, the invariant density of S can be calculated from that of T.



This is the type of auxiliary transformations we shall use to prove the theorem stated at the beginning.

3. Boole's transformation: $T(x) = x - \frac{1}{x}$, $\quad x \in \mathbb{R}$, $\quad \frac{d\mu}{d\lambda} = 1$.



Difficulty: The natural reference measure is infinite; $\pm\infty$ are indifferent fixed points.

Remedy: (i) Change variables to get a map on $[0, 1]$ and define S in a similar way as in the preceding example, or

(ii) *'induce away' from the indifferent fixed points* ([AW]):

$$A = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right] \quad (A = [-1, 1] \text{ in [AW]})$$

$$\varphi_A(x) = \inf\{n \geq 1 : T^n(x) \in A\} \quad \text{(first return time)}$$

$$S(x) = T^{\varphi(x)}(x), \; x \in A \quad \ldots \quad \text{induced or first return map on } A.$$

S has full branches, and satisfies the conditions of R. Adler's *Folklore theorem*, i.e.,

- S is uniformly expanding
- $S''/(S')^2$ is bounded.

According to a general property of induced transformations, λ is also invariant for S.

(1.1.2) The general concept

Let T be a non-singular transformation on the σ-finite measure space $(X, \mathcal{B}, m)$. An auxiliary transformation on a set $A \in \mathcal{B}$ is defined through a function $\varphi : A \to \mathbb{N} \cup \{\infty\}$ satisfying the following conditions:

(i) φ is measurable with respect to $A \cap \mathcal{B}$, and $\varphi < \infty$ a.e. on A,

(ii) $T^{\varphi(x)}(x) \in A, \quad x \in A \cap \{\varphi < \infty\}$.

We call the a.e. defined map

$$S_\varphi : A \to A : S_\varphi(x) = T^{\varphi(x)}(x)$$

the *auxiliary transformation associated with* φ.

Note that (ii) is obviously fulfilled if $A = X$.

<u>Notation</u>:
$$A_n = A \cap \{\varphi = n\}, \quad n \in \mathbb{N},$$
$$D_n = A \cap \{\varphi > n\}, \quad n \in \mathbb{N}_0 .$$

We have:

- $D_0 = A, \quad D_n = A_{n+1} \cup D_{n+1}, \quad n \in \mathbb{N}_0$
- $S_\varphi^{-1}(E) \cap \{\varphi < \infty\} = \bigcup_{n=1}^\infty (A_n \cap T^{-n}E), \; E \subseteq A$

Proposition 0: S_φ is a non-singular transformation on the measure space $(A, A \cap \mathcal{B}, m_{|A\cap\mathcal{B}})$.

Proof: $S_\varphi : A \cap \{\varphi < \infty\} \to A$ is measurable, $m(A \backslash \{\varphi < \infty\}) = 0$, and $m(S_\varphi^{-1}(E)) = 0$ for all $E \in A \cap \mathcal{B}$ with $m(E) = 0$. □

Examples:

1. $A = X$, $\varphi \equiv N$ $(N \in \mathbb{N})$: $S_\varphi = T^N$
2. *Induced transformations* (H. Poincaré, S. Kakutani ([Ka]))
   $A \in \mathcal{B}$, $A \subseteq \bigcup_{n=1}^{\infty} T^{-n}A \pmod m$
   $\varphi(x) = \inf\{n \geq 1 : T^n(x) \in A\} =: \varphi_A(x)$
   $S_\varphi =: T_A$ ... *induced transformation (first return map) on A*
   (see §1.5 in J. Aaronson's book)
3. *Jump transformations* (R. Fischer, 1972 ([F]), F. Schweiger, 1975 ([Sch2]))

   Characteristic: $A = X$, and φ is defined through a partition (see [Sch4]).

   Let $\xi_1 = \{Z_k : k \in I\}$ be a finite or countably infinite measurable partition of X, and let

   $$\xi_n = \bigvee_{j=0}^{n-1} T^{-j}\xi_1 = \{Z_{k_1,\ldots,k_n} : (k_1, \ldots, k_n) \in I^n\},$$
   $$Z_{k_1,\ldots,k_n} = \{x \in X : T^{j-1}(x) \in Z_{k_j},\ 1 \leq j \leq n\}$$
   (cylinder of order n), $n \geq 1$.

   Let ζ be a subclass of $\bigcup_{n=1}^{\infty} \xi_n$ such that

   $$\bigcup_{Z\in\zeta} Z = X \pmod m, \text{ and}$$

   $$\varphi(x) = \inf\{n \geq 1 : x \in Z,\ Z \in \xi_n \cap \zeta\}.$$

   Then S_φ is called the *jump transformation associated with* ζ.

Typical examples are the auxiliary transformations in Example 1 and 2 of (1.1.1).

1. β-transformations:

$$\begin{aligned}\xi_1 &= \{Z_0, Z_1, \dots, Z_{[\beta]}\}\\ \zeta &= \text{class of full cylinders}\end{aligned}$$

($Z \in \xi_n$ is called full if $T^n Z = [0,1[$.) Then $S_\varphi = S$ as in (1.1.1).

2. Daniels–Parry transformation:

$$\xi_1 = \{Z_0, Z_1\}, \quad \zeta = \{Z_{k_1,\dots,k_n} : k_n = 1,\ n \geq 1\}$$
Then $S_\varphi = S$ as in (1.1.1).

In both cases the class ζ coincides with the class of S_φ-cylinders. Typically, jump transformations serve to reduce the class of cylinders to subclasses which

(i) still generate the given σ-algebra (mod m), and

(ii) have specific properties, not valid for the entire cylinder class (full cylinders, bounded distorsion, ...).

(1.1.3) Carrying over ergodic properties from S_φ to T

$S := S_\varphi$ ... auxiliary transformation, $m(A) > 0$.

(A) Conservativity

Lemma: If W is wandering for T, $A \cap W$ is wandering (mod m) for S.

Proof: $\sum_{n=0}^{\infty} 1_{A\cap W} \circ S^n \leq \sum_{n=0}^{\infty} 1_W \circ T^n$ a.e. on A. □

Proposition 1: Suppose that for all $B \in \mathcal{B}$ with $\mu(B) > 0$ there exists an $n = n(B) \geq 0$ such that $m(A \cap T^{-n}B) > 0$. Then,

$$S \text{ conservative} \implies T \text{ conservative.}$$

Proof: If $W \in \mathcal{B}$ is wandering for T, $A \cap T^{-n}W$ is wandering (mod m) for S for all $n \geq 0$. Thus $m(A \cap T^{-n}W) = 0$ for all $n \geq 0$, and so $m(W) = 0$. □

Note that the basic condition is obviously fulfilled if $A = X$.

(B) Ergodicity

Lemma: $E \in \mathcal{B}, T^{-1}E = E \Longrightarrow S^{-1}(A \cap E) = A \cap E \pmod{m}$.

Proof: Use $S^{-1}(E) \cap \{\varphi < \infty\} = \bigcup_{n=1}^{\infty}(A_n \cap T^{-n}E)$. □

Proposition 2: If $\bigcup_{n=0}^{\infty} T^{-n}A = X \pmod{m}$,

$$S \text{ ergodic} \Longrightarrow T \text{ ergodic.}$$

Proof: For $E \in \mathcal{B}$ with $T^{-1}E = E$ we have $m(A \cap E) = 0$ or $m(A \cap E^c) = 0$. Suppose $m(A \cap E) = 0$. Then $m(T^{-n}A \cap E) = m(T^{-n}(A \cap E)) = 0$ for all $n \geq 0$. Thus $m(E) = m\Big(\bigcup_{n=0}^{\infty} T^{-n}A \cap E\Big) = 0$. □

(C) Invariant measures

Let ν be a measure on $A \cap \mathcal{B}$, $\nu \ll m_{|A\cap\mathcal{B}}$. Then

$$\mu(E) = \sum_{n=0}^{\infty} \nu(D_n \cap T^{-n}E),\ E \in \mathcal{B},$$

defines a measure on $\mathcal{B}$, and $\mu \ll m$.

Proposition 3: ν invariant for $S \Longrightarrow \mu$ invariant for T.

Proof: For $E \in \mathcal{B}$,

$$\begin{aligned} \mu(T^{-1}E) &= \sum_{n=0}^{\infty} \nu(D_n \cap T^{-(n+1)}E) \quad [D_n = A_{n+1} \cup D_{n+1}] \\ &= \sum_{n=1}^{\infty} \nu(A_n \cap T^{-n}E) + \sum_{n=1}^{\infty} \mu(D_n \cap T^{-n}E), \text{ and} \end{aligned}$$

$\sum_{n=1}^{\infty} \nu(A_n \cap T^{-n}E) = \nu(S^{-1}(A \cap E)) = \nu(A \cap E) = \nu(D_0 \cap E)$. □

Exercises: **1.** $T(x) = \beta x \pmod{1}$ $(\beta > 1)$ has a finite invariant measure $\mu \sim \lambda$, whose density is given by
$\frac{d\mu}{d\lambda}(x) = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \cdot 1_{[0,T^n 1[}(x),\ x \in [0,1[$ (Gel'fond–Parry formula ([Ge], [P1])).

2. Let

$$T(x) = \begin{cases} \dfrac{x}{1-x}, & x \in [0, 1/2], \\ \dfrac{1}{x} - 1, & x \in (1/2, 1], \end{cases}$$

and let ν be the measure on $[0,1]$ with density $\dfrac{d\nu}{d\lambda}(x) = \dfrac{1}{1+x}$, $x \in [0,1]$. Using the fact that ν is invariant for the continued fraction map $S(x) = \dfrac{1}{x}$ (mod 1), calculate the invariant density for T.

(1.1.4) Remarks on induced transformations

For induced transformations 'converses' of (A), (B) and (C) are available.

Let T_A be as in Example 2 of (1.1.2).

(A)' T conservative $\Longrightarrow$ T_A conservative.

Proof: $E \in A \cap \mathcal{B} : \sum\limits_{n=0}^{\infty} 1_E \circ T^n = \sum\limits_{n=0}^{\infty} 1_E \circ T_A^n$ a.e. on A. □

(B)' T ergodic $\Longrightarrow$ T_A ergodic.

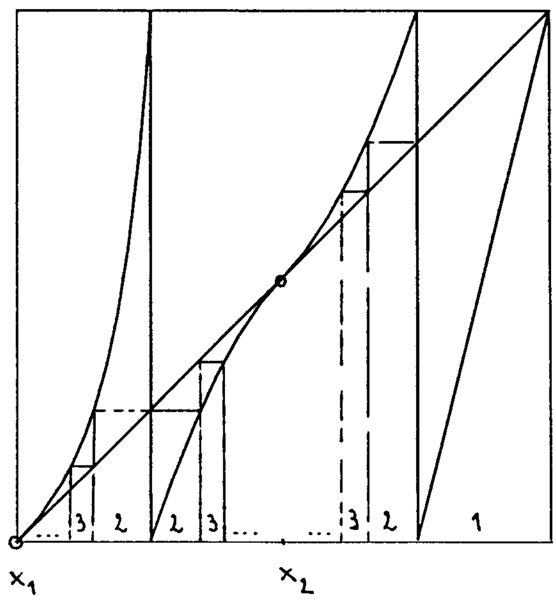
Proof: If $E \in A \cap \mathcal{B}$, $T_A^{-1}(E) = E$, the set
$E \cup \bigcup\limits_{n=1}^{\infty} \left(\bigcap\limits_{j=0}^{n-1} T^{-j} A^c \cap T^{-n} E \right)$ is invariant (mod m) for T. □

(C)' If m is invariant for T and $m(A) < \infty$, $m_{|A \cap \mathcal{B}}$ is invariant for T_A.

Proof: See (1.3) of these notes (or §1.5 in [Aa1]). □

(1.2) Sketch of proof of the basic ergodic properties

Let $T: [0,1] \to [0,1]$ satisfy (T1) - (T5).



Choice of φ:

$\varphi(x) = 1,$

$x \in \bigcup_{k \notin J} Z_k\,;$

$x \in Z_j\,,\ j \in J$:

$\varphi(x) =$

$= 1 + \inf\{n \geq 1 : T^n x \notin Z_j\}$

$(< \infty \text{ if } x \neq x_j)$;

$S = S_\varphi$.

φ is measurable as

$$\{\varphi = 1\} = \bigcup_{k \notin J} Z_k\,,$$

$$\{\varphi = n\} = \bigcup_{j \in J} \bigcup_{k \neq j} Z_{\underbrace{j,\ldots,j}_{n-1},k},\ n \geq 2.$$

S has full branches, and satisfies the following conditions.

Proposition:

(1) $\exists\, \rho > 1$ such that $|S'| \geq \rho$ (S is uniformly expanding)

(2) $S''/(S')^2$ is bounded (Adler's condition).

(1) is immediate. To get (2) we use a kind of generalization of the summation formula for the geometric series, given by the following lemma.

Lemma ([Th1]): Let $f : [0, \eta] \to \mathbb{R}$ ($\eta > 0$) be differentiable, increasing and concave, satisfying $0 < f(x) < x, 0 < x \leq \eta$. Then the following estimates hold for $0 < x \leq \eta$:

$$1 + f'(x)\frac{x}{x - f(x)} \leq \sum_{n=0}^{\infty}(f^n)'(x) \leq \frac{x}{x - f(x)}.$$

In particular, $\sum_{n=0}^{\infty}(f^n)'(x) \sim \dfrac{x}{x - f(x)}$ $(x \to 0)$.

Example: For $f(x) = q \cdot x$ $(0 < q < 1)$ equality holds.

Recall that $u_j = (T_{|Z_j})^{-1}$, which admits a C^1-extension to $[0,1]$ $(j \in J)$. Using the Lemma we see that, for each $j \in J$ and each $\delta > 0$,

$(\star)$ $\sum_{n=0}^{\infty}(u_j^n)'$ is uniformly convergent on $[0,1] \backslash (x_j - \delta, x_j + \delta)$.

In particular, $\sum_{n=0}^{\infty}(u_j^n)'$ is bounded on sets which are bounded away from x_j, and a routine calculation now proves that S satisfies Adler's condition.

As a consequence of the Proposition, S is exact (λ) and has an invariant probability measure $\nu \sim \lambda$, such that $\frac{d\nu}{d\lambda}$ has a version h_S which is Lipschitz continuous and positive on $[0,1]$. From (1.1.3) we get:

T is conservative and ergodic (λ), and

$$\mu(E) = \sum_{n=0}^{\infty} \nu(D_n \cap T^{-n}E), E \in \mathcal{B},$$

defines an invariant measure for T, equivalent to λ.

As $D_n = \bigcup_{j \in J} Z_{\underbrace{j, \ldots, j}_{n}}$ (mod λ), $n \geq 1$, the density of μ has a version given by

$$h = h_S + \sum_{j \in J} \sum_{n=1}^{\infty} h_S \circ u_j^n \cdot (u_j^n)'.$$

$(\star)$ implies that h is continuous on $[0,1] \backslash \{x_j : j \in J\}$, and we have $h \geq h_S > 0$ on $[0,1]$.

Finally, the asymptotic equivalence in the Lemma shows that

$$\lim_{x \to x_i} h(x) \cdot \prod_{j \in J} \frac{x - u_j(x)}{x - x_j} \quad (i \in J)$$

exists and is positive and finite. This establishes the formula for h.

Uniqueness of μ is dicussed in the next section. For the proof of exactness we refer to Theorem 4.4.7 in J. Aaronson's book. $\square$

(1.3) Uniqueness of invariant measures

The aim here is to prove the following theorem employing the technique of induced transformations. Again, a question of infinite ergodic theory will be settled by relating it to finite ergodic theory via auxiliary transformations.

Theorem (Uniqueness of invariant measures)

Let T be a conservative ergodic transformation of a σ-finite measure space $(X, \mathcal{B}, m)$. Then, up to multiplication by constants, there is at most one m-absolutely continuous, σ-finite T-invariant measure.

The proof is based on the representation formula given by the next proposition. As before, φ_A denotes the first return time of A, and T_A the induced transformation on A.

Proposition: Let T be measure preserving on the measure space $(X, \mathcal{B}, \mu)$, and let $A \in \mathcal{B}$ satisfy $\mu(A) < \infty$ and $A \subseteq \bigcup_{n=1}^{\infty} T^{-n}A \pmod{\mu}$. Then,

$$\mu(E) = \sum_{n=0}^{\infty} \mu(A \cap \{\varphi_A > n\} \cap T^{-n}E), \quad E \in \left(\bigcup_{n=0}^{\infty} T^{-n}A\right) \cap \mathcal{B}.$$

Proof: We extend φ_A to X by $\varphi(x) = \inf\{n \geq 1 : T^n x \in A\}$, $x \in X$. Using

$$T^{-1}(A^c \cap \{\varphi > n\}) = (A \cap \{\varphi > n+1\}) \cup (A^c \cap \{\varphi > n+1\}), \; n \geq 0,$$

and the invariance of μ we get by induction

$$\begin{aligned} \mu(E) &= \sum_{k=0}^{n} \mu(A \cap \{\varphi > k\} \cap T^{-k}E) + \\ &\quad + \mu(A^c \cap \{\varphi > n\} \cap T^{-n}E), \; n \geq 0, \; E \in \mathcal{B}. \end{aligned}$$

$\underline{E = T^{-1}A}$: $\mu(A) = \mu(T^{-1}A)$

$$= \sum_{k=0}^{n} \mu(A \cap \{\varphi = k+1\}) + \mu(A^c \cap \{\varphi = n+1\}),\ n \geq 0;$$

since $\mu(A) = \sum_{k=1}^{\infty} \mu(A \cap \{\varphi = k\}) < \infty$, we have

$$\lim_{n\to\infty} \mu(A^c \cap \{\varphi = n\}) = 0.$$

$\underline{E \subseteq A}$: $A \cap \{\varphi > n\} \cap T^{-n}E = \emptyset$, $n \geq 1$, hence the formula is obviously true;

$\underline{E \subseteq A^c \cap \{\varphi = N\}}$ $(N \geq 1)$: $A^c \cap \{\varphi > n\} \cap T^{-n}E \subseteq A^c \cap \{\varphi = n + N\}$ $(n \geq 0)$, so $\lim_{n\to\infty} \mu(A^c \cap \{\varphi > n\} \cap T^{-n}E) = 0$, i.e., the formula holds.

Therefore the formula holds for all measurable $E \subseteq \bigcup_{n=0}^{\infty} T^{-n}A$. □

Corollary 1: Let the conditions of the Proposition be satisfied. Then,

(i) $\mu_{|A\cap\mathcal{B}}$ is invariant for T_A, and

(ii) the following useful identities hold:

$$\mu(A \cap \{\varphi > n\}) = \mu(A^c \cap \{\varphi = n\}),\ n \in \mathbb{N}$$

$(\varphi(x) = \inf\{n \geq 1 : T^n(x) \in A\},\ x \in X)$.

Proof: (i) $E \in A \cap \mathcal{B}$:

$$\begin{aligned} \mu(E) &= \mu(A \cap E) = \mu(T^{-1}(A \cap E)) \\ &= \sum_{n=0}^{\infty} \mu(A \cap \{\varphi > n\} \cap T^{-(n+1)}A \cap T^{-(n+1)}E) \\ &= \sum_{n=0}^{\infty} \mu(A \cap \{\varphi = n+1\} \cap T^{-(n+1)}E) = \mu(T_A^{-1}E). \end{aligned}$$

(ii) In the preceding proof we found

$$\mu(A) = \mu(A \cap \{\varphi \leq n\}) + \mu(A^c \cap \{\varphi = n\}),\ \ n \geq 1,$$

which gives (ii). □

Corollary 2: Let T be conservative, ergodic and measure preserving on the σ-finite measure space $(X, \mathcal{B}, \mu)$. Then, for all $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$,

(i) $\mu(E) = \sum_{n=0}^{\infty} \mu(A \cap \{\varphi_A > n\} \cap T^{-n}E), \quad E \in \mathcal{B}$, and

(ii) $\mu(X) = \int_A \varphi_A \, d\mu$ (Kac's formula).

Proof: (i) Since T is conservative and ergodic, $\bigcup_{n=0}^{\infty} T^{-n}A = X \pmod{\mu}$ for each $A \in \mathcal{B}$ with $\mu(A) > 0$, and the Proposition applies.

(ii) If we put $E = X$ in (i), we obtain

$$\mu(X) = \sum_{n=0}^{\infty} \mu(A \cap \{\varphi_A > n\}) = \int_A \varphi_A \, d\mu. \qquad \square$$

Remarks:

1. (i) shows that μ is determined by its values on $A \cap \mathcal{B}$ for arbitrary sets $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$. (The formula actually also holds if $\mu(A) = \infty$.)

2. (ii) says that *infinite ergodic theory* deals with dynamical systems with *infinite mean return times.* In fact, under the conditions of Corollary 2, $\mu(X) = \infty \iff \int_A \varphi_A \, d\mu = \infty$ for one, and hence for all, $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$.

Proof of the Uniqueness Theorem

Let $\mu, \nu \ll m$ be T-invariant σ-finite measures (with $\mu(X), \nu(X) > 0$). Then $\mu, \nu \sim m$. To see this, let $B \in \mathcal{B}$ with $m(B) > 0$. Since T is conservative and ergodic with respect to m, $\bigcup_{n=0}^{\infty} T^{-n}B = X \pmod{m}$, and therefore also $\pmod{\mu}$ and $\pmod{\nu}$. In particular, $\mu(B), \nu(B) > 0$.

Now choose $A \in \mathcal{B}$ such that $0 < \mu(A), \nu(A) < \infty$. We may assume $\mu(A) = \nu(A) = 1$. Then $\mu_{|A \cap \mathcal{B}}, \nu_{|A \cap \mathcal{B}}$ are equivalent ergodic invariant probability measures for T_A. As we know from finite ergodic theory, this implies $\mu = \nu$ on $A \cap \mathcal{B}$, and the formula in Corollary 2 (i) yields $\mu = \nu$. ∎

References

[Aa1] J. Aaronson, *An introduction to infinite ergodic theory,* Amer. Math. Soc., 1997

[Aa2] J. Aaronson, *Ergodic theory for inner functions of the upper half plane,* Ann. Inst. Henri Poincaré, Sec. B, **14** (1978), 233-253

[Ad] R. L. Adler, *F-expansions revisited,* Recent Advances in Topological Dynamics, LNM **318**, Springer, 1973, pp.1-5

[AW] R. L. Adler and B. Weiss, *The ergodic infinite measure preserving transformation of Boole,* Isr. J. Math. **16** (1973), 263 - 278

[Boo] G. Boole, *On the comparison of transcendents with certain applications to the theory of definite integrals,* Philos. Trans. Roy. Soc. London, **147**, Part III (1857), 745 - 803

[Bow] R. Bowen, *Invariant measures for Markov maps of the interval,* Commun. Math. Phys. **69** (1979), 1 - 17

[D] H. E. Daniels, *Processes generating permutation expansions,* Biometrika **49** (1962), 139 - 149

[F] R. Fischer, *Ergodische Theorie von Ziffernentwicklungen in Wahrscheinlichkeitsräumen,* Math. Z. **128** (1972), 217 - 230

[Ge] A. O. Gel'fond, *A common property of number systems* (in Russian), Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 809 - 814

[Gl] J. W. L. Glaisher, *On a theorem in definite integration,* Quart. J. Pure Appl. Math. **10** (1870), 347 - 356

[Ka] S. Kakutani, *Induced measure preserving transformations,* Proc. Imp. Acad. Sci. Tokyo **19** (1943), 635 - 641

[Ko] R. Kołodziej, *An infinite smooth invariant measure for some transformation of a circle,* Bull. Acad. Polon. Sci. Ser. Sci. Math. **29** (1981), 549 - 551

[Ma] P. Manneville, *Intermittency, self-similarity and 1/f spectrum in dissipative dynamical systems,* J. Phys. (Paris) **41** (1980), 1235 - 1243

[Mi] M. Misiurewicz, *The Result of Rafał Kołodziej* , Monographie N°29 de L'Enseignement Mathématique (Théorie Ergodique), Université de Genève, 1981

[P1] W. Parry, *On $\beta-$expansions of real numbers,* Acta Math. Acad. Sci. Hung. **11** (1960), 401 - 416

[P2] —, *Ergodic properties of some permutation processes,* Biometrika **49** (1962), 151 - 154

[R] A. Rényi, *Representations for real numbers and their ergodic properties,* Acta Math. Acad. Sci. Hung. **8** (1957), 477 - 493

[Sch1] F. Schweiger, *Numbertheoretical endomorhisms with $\sigma-$finite invariant measure,* Isr. J. Math. **21** (1975), 308 - 318

[Sch2] —, *Some remarks on ergodicity and invariant measures,* Michigan Math. J. **22** (1975), 181 - 187

[Sch3] —, $\tan x$ *is ergodic,* Proc. Amer. Math. Soc. **71** (1978), 54 - 56

[Sch4] —, *Ergodic theory of fibered systems and metric number theory,* Clarendon Press, Oxford, 1995

[Th1] M. Thaler, *Estimates of the invariant densities of endomorphisms with indifferent fixed points,* Isr. J. Math. **37** (1980), 303 - 314

[Th2] —, *Transformations on* [0,1] *with infinite invariant measures,* Isr. J. Math. **46** (1983), 67 - 96

[Th3] —, *The invariant densities for maps modeling intermittency,* J. Statist. Phys. **79** (1995), 739 - 741

[Z1] R. Zweimüller, *Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points,* Nonlinearity **11** (1998), 1263 - 1276

[Z2] —, *Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points,* Ergod. Th. & Dynam. Sys. **20** (2000), 1519 - 1549

Infinite ergodic theory: One-dimensional maps with indifferent fixed points

Maximilian Thaler (Salzburg)

2. Iteration of the Perron–Frobenius operator

The purpose of part 2 is to present a result on the asymptotic behaviour of the Perron–Frobenius operator for the maps with indifferent fixed points specified at the beginning, and to introduce several basic concepts from general infinite ergodic theory connected with this result.

(2.1) The Perron–Frobenius operator ([Aa1], §1.3)

Let T be a non-singular transformation on the σ-finite measure space $(X, \mathcal{B}, m)$.

For $f \in L_1(m)$ let ν_f denote the measure with density f. As T is non-singular, $\nu_f \circ T^{-1} \ll m$, and the Radon–Nikodym derivative

$$\frac{d(\nu_f \circ T^{-1})}{dm} =: Pf \qquad \text{exists.}$$

P is a positive linear operator on $L_1(m)$, characterized by

$$\int_A Pf\, dm = \int_{T^{-1}A} f\, dm, \quad f \in L_1(m), \quad A \in \mathcal{B}.$$

An approximation argument shows that

$$\int Pf \cdot g\, dm = \int f \cdot g \circ T\, dm, \quad f \in L_1(m), \quad g \in L_\infty(m).$$

The definition of P extends naturally to the set of all non-negative measurable functions. P is called the *Perron–Frobenius operator* (dual, transfer, Kuzmin, ... operator) of T with respect to m.

Probabilistic meaning:

f ... probability density

X_0 ... random variable with values in X and density f (initial value of the iteration process)

$\Longrightarrow$ $X_n := T^n(X_0)$ (position after n steps) has density $P^n f$:

$$\text{Prob}\,(X_n \in A) = \int_A P^n f\, dm, \quad A \in \mathcal{B}, \quad n \geq 0.$$

Basic ergodic properties of T in terms of P:

1. T conservative $\iff \exists\, f \in L_1(m),\ f \geq 0: \ \sum_{n=0}^{\infty} P^n f = \infty$ a.e.

   $\iff \forall\, f \in L_1(m),\ f > 0$ a.e. $: \ \sum_{n=0}^{\infty} P^n f = \infty$ a.e.

2. T conservative and ergodic $\iff$

   $\forall\, f \in L_1(m),\ f \geq 0,\ \int f\, dm > 0: \ \sum_{n=0}^{\infty} P^n f = \infty$ a.e.

3. T exact $\iff \forall\, f \in L_1(m),\ \int f\, dm = 0: \ \|P^n f\|_1 \to 0$.

4. If μ is a measure on $\mathcal{B}$ with density h,

   $$\mu \text{ invariant for } T \iff Ph = h.$$

(2.2) A convergence theorem

Whereas for interval maps with indifferent fixed points and finite λ-absolutely continuous invariant measure strong results on the iteration of the Perron–Frobenius operator are available (see e.g. [FS], [LSV], [Yo]), only basic qualitative results have been proved for the infinite measure case.

Let $(X, \mathcal{B}, m) = ([0,1], \mathcal{B}, \lambda)$, and let T satisfy (T1) - (T5). As in part 1, μ denotes the λ-absolutely continuous invariant measure, and h a version of $\frac{d\mu}{d\lambda}$ which is continuous on $[0,1] \setminus \{x_j : j \in J\}$.

Let $\xi_n = \bigvee_{j=0}^{n-1} T^{-j} \xi_1$ be the class of cylinders of order n $(n \geq 1)$. For $Z \in \xi_n$, let

$$u_Z := (T^n|_Z)^{-1}, \text{ extended to a } C^1\text{-function on } [0,1].$$

Then,

$$P^n f = \sum_{Z \in \xi_n} f \circ u_Z \cdot |u_Z'|, \quad f \in L_1(\lambda) \text{ or } f \geq 0 \text{ measurable}, \ n \geq 1.$$

Throughout, $P^n f$ denotes the version given by this formula.

As T has no λ-absolutely continuous invariant probability measure, Krengel's stochastic ergodic theorem ([Kr], [Aa1]) tells us that

$$\frac{1}{n} \sum_{k=0}^{n-1} P^k f \xrightarrow{\lambda} 0 \quad \text{for all} \quad f \in L_1(\lambda).$$

However, proper normalization leads to non-trivial limiting behaviour.

Theorem 1 ([CF], [Th4], [Z2]): There exists a sequence (a_n) of positive numbers such that for all Riemann-integrable f on $[0,1]$

$$\frac{1}{a_n}\sum_{k=0}^{n-1} P^k f \;\to\; \left(\int f\,d\lambda\right) h$$

uniformly on compact subsets of $[0,1]\backslash\{x_j : j \in J\}$.

If $T(x) = x \pm a_j|x - x_j|^{p_j+1} + o(|x - x_j|^{p_j+1})(x \to x_j)$ with $a_j > 0$, $p_j \geq 1$ $(j \in J)$, and $p = \max\{p_j : j \in J\}$,

$$a_n \sim \text{const.} \begin{cases} \dfrac{n}{\log n}, & p = 1, \\[2ex] n^{1/p}, & p > 1. \end{cases}$$

(The proof will be sketched in (2.3) and (2.4).)

Let $\hat{T}$ denote the Perron–Frobenius operator of T with respect to the invariant measure μ. As

$$\hat{T}f = \frac{1}{h}P(fh), \quad f \in L_1(\mu),$$

Theorem 1 asserts that

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\hat{T}^k f \;\to\; \int f\,d\mu$$

uniformly on compact subsets of $[0,1]\backslash\{x_j : j \in J\}$ for all $f \in L_1(\mu)$ such that fh is R-integrable on $[0,1]$. This means that T is pointwise dual ergodic, and has large classes of uniform sets and Darling–Kac sets. We recall these concepts, which are particularly relevant to questions of distributional convergence ([Aa1], §§3.7, 3.8).

Let T be conservative, ergodic and measure preserving on the σ-finite measure space $(X, \mathcal{B}, \mu)$, and let $\hat{T}$ denote the Perron–Frobenius operator of T.

T is called *pointwise dual ergodic,* if there exists a sequence (a_n) of positive numbers such that

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\hat{T}^k f \;\to\; \int f\,d\mu \text{ a.e. for all } f\in L_1(\mu).$$

The sequence (a_n) is called the *return sequence of* T. ($\lim_{n\to\infty} a_n = \infty$, and, if $\mu(X)=\infty$, $\lim_{n\to\infty}\frac{a_n}{n}=0$.)

A set $A\in\mathcal{B}$ with $0<\mu(A)<\infty$ is called

- *uniform for* $f\in L_1(\mu)_+$ $(=\{f\in L_1(\mu):\ f\geq 0,\ \int f\,d\mu>0\})$ if there exists a sequence (a_n) of positive numbers such that

  $$\frac{1}{a_n}\sum_{k=0}^{n-1}\hat{T}^k f \;\to\; \int f\,d\mu \text{ almost uniformly on } A$$

  (i.e., in $L_\infty(\mu|_{A\cap\mathcal{B}})$),
- *a unifom set* if A is uniform for some $f\in L_1(\mu)_+$,
- *a Darling–Kac set* if A is uniform for $f=1_A$.

We mention the following

Proposition (see Proposition 3.7.5 in [Aa1]): Let (a_n) be a sequence in $\mathbb{R}^+$. If, for some $f\in L_1(\mu)_+$ and some $c\in\mathbb{R}^+$,

$$\frac{1}{a_n}\sum_{k=0}^{n-1}\hat{T}^k f \;\to\; c \quad \text{a.e.}$$

on a set of positive measure, then T is pointwise dual ergodic.

(The proof rests on the representation formula for μ in (1.3) and Hurewicz's ergodic theorem.)

Thus, in view of Egorov's theorem we have:

$$T \text{ pointwise dual ergodic} \iff T \text{ has uniform sets.}$$

Now we return to our specific situation.

Corollary: Let $T : [0,1] \to [0,1]$ satisfy (T1) - (T5). Any set $A \in \mathcal{B}$ with $\lambda(A) > 0$ which is bounded away from the indifferent fixed points is a uniform set for T. If, in addition, $\lambda(\partial A) = 0$ (i.e. A is a continuity set for λ), A is a Darling–Kac set for T. □

Remark: By a result in [Z2], every set $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ can be approximated arbitrarily close both from the inside and from the outside by measurable sets which are not Darling–Kac sets.

(2.3) Establishing convergence: Outline of the main steps

The key to prove an ergodic theoretic result for the maps in our class usually is to establish an appropriate lemma concerning the iteration of a function near an indifferent fixed point. An example is the Lemma in (1.2), which we used to prove the basic ergodic properties. As a particular consequence we noted that for each $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\sum_{k=0}^{\infty} (u_j^k)' \;\leq\; C(\varepsilon) \;\text{ on }\; [0,1] \backslash \bigcup_{j \in J} (x_j - \varepsilon,\, x_j + \varepsilon)$$

for all $j \in J$. The nucleus of the proof of the present result is a stronger version of these estimates.

Notation:

- $u_{k_1,\ldots,k_n} := u_Z\,, \quad Z = Z_{k_1,\ldots,k_n} \in \xi_n \quad ((k_1,\ldots,k_n) \in I^n,\; n \geq 1)$
- $A_\varepsilon := [0,1] \backslash \bigcup_{j \in J} (x_j - \varepsilon,\, x_j + \varepsilon),\; \varepsilon > 0.$

Lemma: For each $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for all $n \geq 1$ and <u>all</u> $(k_1,\ldots,k_n) \in I^n$,

$$\sum_{i=1}^{n} |u'_{k_i,\ldots,k_n}| \;\leq\; C(\varepsilon) \;\text{ on }\; A_\varepsilon\,.$$

$$\Big(\text{If }\; (k_1,\ldots,k_n) = (j,\ldots,j), \quad \sum_{i=1}^{n} |u'_{k_i,\ldots,k_n}| \;=\; \sum_{k=1}^{n} (u_j^k)'.\Big)$$

To prove the convergence theorem, it suffices to do the main work for functions f satisfying:

($\star$) f continuous and positive on $[0,1]$, and differentiable on $(0,1)$ with bounded derivative.

The rest is settled by an approximation procedure.

Using the above lemma we get the following

Proposition: Let f satisfy ($\star$). Then,

(i) $P^n f$ satisfies ($\star$) for each $n \geq 0$, and

(ii) for any $\varepsilon > 0$ there exists a constant $K = K(f, \varepsilon)$ such that

$$|(P^n f)'| \leq K \cdot P^n f \quad \text{on } A_\varepsilon .$$

Assume now f satisfies ($\star$), and let

$$f_n := \left(\sum_{k=0}^{n-1} P^k f \right) \Big/ \left(\sum_{k=0}^{n-1} (P^k f)(t_0) \right), \; n \geq 1,$$

for a fixed $t_0 \in [0,1] \backslash \{x_j : j \in J\}$. By the Proposition, (f_n) is uniformly Lipschitz and bounded on A_ε for each $\varepsilon > 0$. Exploiting the uniqueness of the invariant measure we get by means of the Arzelà–Ascoli theorem:

$$h(t_0)\, f_n \;\rightarrow\; h \;\text{ uniformly on } A_\varepsilon \text{ for each } \varepsilon > 0.$$

Finally, Hurewicz's ergodic theorem shows that the normalizing sequence does not depend on f. □

(2.4) Identifying return sequences: The asymptotic renewal equation

The purpose of this section is to explain how to determine the return sequence (a_n) for those maps in our class which behave 'regularly' at the indifferent fixed points, e.g., in the sense of the second part of Theorem 1. The main tool is Karamata's Tauberian theorem, in combination with the asymptotic renewal equation ([Aa1], §3.8).

We consider the general setting: T conservative, ergodic and measure preserving on the σ-finite measure space $(X, \mathcal{B}, \mu)$ with $\mu(X) = \infty$, and pointwise dual ergodic with return sequence (a_n).

Throughout, let $A \in \mathcal{B}$, $0 < \mu(A) < \infty$, be a given uniform set, and let f be a probability density such that A is uniform for f, i.e.,

$$\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f \;\to\; 1 \text{ almost uniformly on } A.$$

The probability measure with density f is denoted by ν. Integration yields

$$\mu(A)\, a_n \sim \sum_{k=0}^{n-1} \nu(T^{-k} A) \quad (n \to \infty).$$

Interpreting the successive visits to A as a (delayed) renewal process with initial distribution ν, the sequence $(\nu(T^{-n}(A))$ is the associated renewal sequence.

To determine (a_n) we shall essentially proceed as in the classical case. Suppose $(u_n)_{n=0}^{\infty}$ is the renewal sequence associated with the probability distribution $(p_n)_{n=1}^{\infty}$ on $\mathbb{N}$, i.e.,

$$u_0 = 1, \;\; u_n = p_1 u_{n-1} + \ldots + p_n u_0 \,, \;\; n \geq 1.$$

Let $q_n = \sum_{k>n} p_k$, $n \geq 0$, and let U, F, Q denote the Laplace transforms of $(u_n), (p_n), (q_n)$ respectively, i.e.,

$$U(s) = \sum_{n=0}^{\infty} u_n\, e^{-ns}, \;\; F(s) = \sum_{n=1}^{\infty} p_n\, e^{-ns}, \;\; Q(s) = \sum_{n=0}^{\infty} q_n\, e^{-ns}, \;\; s > 0.$$

The recursion formula for (u_n) is equivalent to

$$U(s)(1 - F(s)) = 1, \;\; s > 0.$$

Since $1 - F(s) = (1 - e^{-s}).\, Q(s)$, $s > 0$, this is the same as

$$U(s) \cdot Q(s) = \frac{1}{1 - e^{-s}}\,, \;\; s > 0,$$

which connects the order of $\left(\sum_{k=0}^{n-1} u_k\right)$ with that of $\left(\sum_{k=0}^{n-1} q_k\right)$. The relevant point is that

$$U(s) \cdot Q(s) \sim \frac{1}{s} \quad (s \to 0),$$

and this relation carries over to our situation.

Asymptotic renewal equation:
Let φ be the first return time of A, and let U, Q be defined by

$$U(s) = \sum_{n=0}^{\infty} \nu(T^{-n}A)\, e^{-ns}, \quad Q(s) = \sum_{n=0}^{\infty} \frac{\mu(A \cap \{\varphi > n\})}{\mu(A)}\, e^{-ns}, \ s > 0.$$

Then,

$$U(s) \cdot Q(s) \sim \frac{1}{s} \quad (s \to 0).$$

Proof: Since

$$A_n := \bigcup_{k=0}^{n} T^{-k}A = \bigcup_{k=0}^{n} T^{-k}(A \cap \{\varphi > n-k\}),$$

and the sets $T^{-k}(A \cap \{\varphi > n-k\})$, $0 \le k \le n$, are disjoint, we have

$$\nu(A_n) = \int_A \sum_{k=0}^{n} \hat{T}^k f \cdot 1_{A\cap\{\varphi > n-k\}}\, d\mu, \quad n \ge 0.$$

Thus,

$$(\star) \quad \int_A \left(\sum_{n=0}^{\infty} \hat{T}^n f\, e^{-ns}\right)\left(\sum_{n=0}^{\infty} 1_{A\cap\{\varphi>n\}}\, e^{-ns}\right) d\mu = \sum_{n=0}^{\infty} \nu(A_n)\, e^{-ns}, \ s > 0.$$

As

$$\sum_{k=0}^{n} \hat{T}^k f \sim \frac{1}{\mu(A)} \sum_{k=0}^{n} \nu(T^{-k}A) \quad (n \to \infty)$$

almost uniformly on A, it is not difficult to see that

$$\sum_{n=0}^{\infty} \hat{T}^n f\, e^{-ns} \sim \frac{1}{\mu(A)}\, U(s) \quad (s \to 0) \quad \text{almost uniformly on } A.$$

Therefore, $(\star)$ yields

$$U(s) \cdot Q(s) \sim \sum_{n=0}^{\infty} \nu(A_n)\, e^{-ns} \quad (s \to 0).$$

Finally, since $\lim_{n\to\infty} \nu(A_n) = 1$,

$$\sum_{n=0}^{\infty} \nu(A_n)\, e^{-ns} \sim \frac{1}{1-e^{-s}} \sim \frac{1}{s} \quad (s \to 0). \qquad \square$$

As mentioned above, the main analytic tool we need is Karamata's Tauberian theorem, a core result from the theory of regular variation (see [BGT] for a comprehensive account). We recall the concepts of regularly varying functions and sequences.

- A measurable function $L : \mathbb{R}^+ \to \mathbb{R}$ with $L > 0$ on (a, ∞) for some $a > 0$ is called *slowly varying at* ∞ if

  $$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1 \quad \text{for all } c > 0$$

  (e.g. $L(x) \equiv L > 0$, $L(x) = \log x, \dots$).

- A function $f : \mathbb{R}^+ \to \mathbb{R}$ is called *regularly varying at* ∞ *with index (exponent)* ρ $(\rho \in \mathbb{R})$ if

  $$f(x) = x^\rho \cdot L(x),\ x \in \mathbb{R}^+,$$

  with L is slowly varying at ∞, or, equivalently, if f is measurable, $f > 0$ on (a, ∞) for some $a > 0$, and

  $$\lim_{x\to\infty} \frac{f(cx)}{f(x)} = c^\rho \quad \text{for all } c > 0.$$

- A sequence (b_n) is regularly varying with index (exponent) ρ if $b_n = f(n)$, $n \geq 1$, where $f : \mathbb{R}^+ \to \mathbb{R}$ is regularly varying at ∞ with index ρ.

The following result is taken from [Fe2], p. 447.

Karamata's Tauberian theorem for power series (KTT):

Let $b_n \geq 0 \quad (n \geq 0)$, and suppose that

$$B(s) = \sum_{n=0}^{\infty} b_n \, e^{-ns}$$

converges for $s > 0$. If L varies slowly at ∞ and $0 \leq \rho < \infty$, then each of the following two relations implies the other:

(1) $B(s) \sim \left(\frac{1}{s}\right)^{\rho} \cdot L\left(\frac{1}{s}\right) \quad (s \to 0)$ and

(2) $\sum_{k=0}^{n-1} b_k \sim n^{\rho} \cdot L(n)/\Gamma(1+\rho) \quad (n \to \infty)$.

Furthermore, if the sequence (b_n) is monotonic and $0 < \rho < \infty$, then (1) is equivalent to

(3) $b_n \sim n^{\rho-1} \cdot L(n)/\Gamma(\rho) \; (n \to \infty)$. □

To sum up, (a_n) can be calculated if we succeed in determining the sequence $(w_n(A))$ given by

$$\underline{w_n(A)} := \sum_{k=0}^{n-1} \mu(A \cap \{\varphi > k\}) = \int_A \min(\varphi, n) \, d\mu, \; n \geq 1,$$

provided that this sequence is regularly varying.

Using (ii) in Corollary 1 in (1.3) we see that

$$w_n(A) = \mu\left(\bigcup_{k=0}^{n-1} T^{-k} A\right), \; n \geq 1.$$

The (order of the) sequence $(w_n(A))$ is called the *wandering rate of* A.

As for the sequence (a_n) we have

$$\frac{1}{n} \, w_n(A) \to 0, \quad \text{and} \quad \lim_{n \to \infty} w_n(A) = \infty \quad (= \mu(X)).$$

Note that, if (a_n) $[(w_n(A))]$ is regularly varying with index α, then $\alpha \in [0, 1]$.

Theorem 2 ([Aa3]): (a_n) is regularly varying with index α if and only if $(w_n(A))$ is regularly varying with index $1-\alpha$. In this case,

$$a_n\, w_n(A) \;\sim\; \frac{n}{\Gamma(1+\alpha)\,\Gamma(2-\alpha)} \quad (n \to \infty).$$

Proof: KTT combined with the asymptotic renewal equation. □

Corollary: If (a_n) $[(w_n(A))]$ is regularly varying,

$$w_n(B) \sim w_n(A) \quad (n \to \infty)$$

for all uniform sets B. □

As a first orientation regarding the asymptotic behaviour of $(a_n\, w_n(A))$ without the assumption of regular variation we mention that

$$1 \le \underline{\lim}\frac{a_n\, w_n(A)}{n}, \quad \overline{\lim}\frac{a_n\, w_n(A)}{n} \le 2$$

always hold. (See Lemma 3.8.5 in [Aa1].)

Proof: Since A is uniform for f we have

$$b_n := \int_A \left(\sum_{k=0}^{n} \hat{T}^k f\right)\left(\sum_{k=0}^{n} 1_{A\cap\{\varphi>k\}}\right) d\mu \sim a_n\, w_n(A) \quad (n \to \infty).$$

On the other hand, the formula

$$\nu(A_n) = \int_A \sum_{k=0}^{n} \hat{T}^k f \cdot 1_{A\cap\{\varphi>n-k\}}\, d\mu \quad (n \ge 0),$$

employed to prove the asymptotic renewal equation, shows that

$$\sum_{k=0}^{n} \nu(A_k) \;\le\; b_n \;\le\; \sum_{k=0}^{2n} \nu(A_k),\ n \ge 0.$$

As $\sum_{k=0}^{n} \nu(A_k) \sim n \quad (n \to \infty)$ the estimates follow. □

Finally, we return to our maps on $[0,1]$. Although the indifferent fixed points are assumed to be regular sources, the sequences $(a_n), (w_n(A))$ are not regularly varying in general. The following result is true, however, for all maps in the class.

Theorem 3 ([Th2]): If $T:[0,1]\longrightarrow[0,1]$ satisfies (T1) - (T5) and A,B are measurable sets of positive measure bounded away from the indifferent fixed points, then

$$w_n(A) \sim w_n(B) \quad (n\longrightarrow\infty).$$

(If $(w_n(A))$ is regularly varying for some A of the above type, the assertion follows from the last corollary. The proof for the general case is too technical to be sketched here.)

Now assume that T satisfies (T1) - (T5), and

$$T(x) = x \pm a_j|x-x_j|^{p_j+1} + o(|x-x_j|^{p_j+1}) \quad (x\longrightarrow x_j)$$

with $a_j>0$, $p_j\geq 1$ $(j\in J)$. Let $p=\max\{p_j : j\in J\}$.

Choosing a suitable set A, e.g. $A=[0,1]\backslash\bigcup_{j\in J} Z_j\cap T^{-1}Z_j$, it is an advanced exercise to show that

$$w_n(A) \sim \text{ const. } \begin{cases} \log n, & p=1, \\ n^{1-\frac{1}{p}}, & p>1, \end{cases}$$

and thus Theorem 2 yields the second part of Theorem 1. □

(2.5) A remark on strong ratio limit theorems

Let $T:[0,1]\longrightarrow[0,1]$ satisfy (T1) - (T5). In view of Theorem 1 a natural question is, whether there exists a sequence (b_n) of positive numbers such that

$$b_n\, P^n f \longrightarrow \left(\int f\, d\lambda\right) h \quad (n\longrightarrow\infty)$$

uniformly on compact subsets of $[0,1]\backslash\{x_j : j\in J\}$ for suitable $f\in L_1(\lambda)$.

As T is exact with respect to λ,

$$P^n f \xrightarrow{\lambda} 0 \quad \text{for all} \quad f\in L_1(\lambda).$$

Proof: Let $f\in L_1(\lambda)$ be non-negative. We show that

$$\lim_{n\to\infty}\int_A P^n f\, d\lambda = 0 \quad \text{for all} \quad A\in\mathcal{B} \quad \text{with} \quad \mu(A)<\infty,$$

which implies convergence to 0 in measure. Let A be given, let B be a measurable set with $0 < \mu(B) < \infty$, and let $g = \frac{\int f\,d\lambda}{\mu(B)} \cdot h\,1_B$. Then,

$$\int_A P^n f\,d\lambda \;\le\; \|P^n f - P^n g\|_1 + \left(\int f\,d\lambda\right) \frac{\mu(B \cap T^{-n}A)}{\mu(B)}, \quad \text{and}$$

$$\mu(B \cap T^{-n}A) \le \mu(T^{-n}A) = \mu(A), \quad n \ge 0.$$

Since T is exact and $\int (f-g)\,d\lambda = 0$,

$$\lim_{n\to\infty} \|P^n f - P^n g\|_1 = 0, \quad \text{and thus} \quad \overline{\lim} \int_A P^n f\,d\lambda \le \left(\int f\,d\lambda\right) \frac{\mu(A)}{\mu(B)}.$$

As $\mu(B)$ can be chosen arbitrarily large, the proof is complete. □

No general result answering the above question for the entire class of interval maps considered here seems to be known. For specific examples a result of this type is proved in [Th5]. We only mention the Lasota–Yorke map, for which $P^n f \xrightarrow{\lambda} 0$ $(f \in L_1(\lambda))$ was first proved in [LY] :

$$T(x) = \begin{cases} \dfrac{x}{1-x}, & x \in \left[0, \dfrac{1}{2}\right] \\[2ex] 2x - 1, & x \in \left(\dfrac{1}{2}, 1\right], \end{cases}$$

with $h(x) = \dfrac{1}{x}$. For this map,

$$(\log n)\, P^n f \to \left(\int f\,d\lambda\right) h \quad (n \to \infty)$$

uniformly on compact subsets of $(0,1]$ for all R-integrable $f : [0,1] \to \mathbb{R}$.

For a discussion of this problem proposing an approach different from that in [Th5] we refer to [Z3], or rather, its author.

References

[Aa1] J. Aaronson, *An introduction to infinite ergodic theory,* Amer. Math. Soc., 1997

[Aa3] J. Aaronson, *Random f-expansions,* Ann. Probab. **14** (1986), 1037 - 1057

[BGT] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation,* Cambridge University Press, Cambridge, 1987

[CF] P. Collet and P. Ferrero, *Some limit ratio theorem related to a real endomorphism in case of a neutral fixed point,* Ann. Inst. H. Poincaré, Phys. Théor. **52** (1990), 283 - 301

[Fe2] W. Feller, *An introduction to probability theory and its applications, Vol. II.* John Wiley & Sons, New York, 1971

[FS] P. Ferrero and B. Schmitt, *On the rate of convergence for some limit ratio theorem related to endomorphisms with a nonregular invariant density,* Manuscript, 1992 (?)

[Kr] U. Krengel, *Ergodic theorems,* W. de Gruyter, Berlin, 1985

[LSV] C. Liverani, B. Saussol, and S. Vaienti, *A probabilistic approach to intermittency,* Ergod. Th. & Dynam. Sys. **19** (1999), 671 - 685

[LY] A. Lasota and J.A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations.* Trans. Amer. Math. Soc. **186** (1973), 481 - 488

[Th2] M. Thaler, *Transformations on* [0,1] *with infinite invariant measures,* Isr. J. Math. **46** (1983), 67 - 96

[Th4] —, *A limit theorem for the Perron-Frobenius operator of transformations on* [0,1] *with indifferent fixed points,* Isr. J. Math. **91** (1995), 111 - 127

[Th5] —, *The asymptotics of the Perron-Frobenius operator for a class of interval maps preserving infinite measures,* Studia Math. **143**(2) (2000), 103 - 119

[Yo] L.-S. Young, *Recurrence times and rates of mixing,* Isr. J. Math. **110** (1999), 153 - 188

[Z2] R. Zweimüller, *Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points,* Ergod. Th. & Dynam. Sys. **20** (2000), 1519 - 1549

[Z3] —, *Ergodic theory of one-dimensional dynamical systems with indifferent fixed points,* PhD Thesis, Erlangen, 2000

Infinite ergodic theory: One-dimensional maps with indifferent fixed points

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3. Distributional limit theorems

In this part we present two examples of distributional limit theorems for infinite measure preserving transformations, which apply to those maps in our class whose return sequence is regularly varying:

- the Darling–Kac theorem, and
- the Dynkin–Lamperti arc-sine laws.

The material of (3.1) and (3.2) can be found in [Aa1], §§3.6, 3.7. For a presentation including the necessary background from advanced analysis and probability we refer to [Z4]. Section (3.3) follows [Th6].

(3.1) J. Aaronson's compactness theorem for distributional convergence

An important question when studying convergence in distribution for sequences defined in terms of a non-singular transformation is to what extent the limiting behaviour depends on the initial distribution. In the situations we shall consider here a compactness theorem due to J. Aaronson enables the conclusion that convergence with respect to one absolutely continuous probability measure implies the same limiting behaviour with respect to all absolutely continuous probabilities.

Notation: Let $(X, \mathcal{B}, m)$ be a measure space, let $f_n : X \to [-\infty, \infty]$ be measurable ($n \geq 1$), and let Y be a random variable on $[-\infty, \infty]$ (i.e., with values in $[-\infty, \infty]$).

If ν is a probability measure on $\mathcal{B}$, we define

$$\underline{f_n \xrightarrow{\nu} Y} :\Longleftrightarrow f_n \xrightarrow{\text{dist}} Y \text{ on the probability space } (X, \mathcal{B}, \nu), \text{ i.e.,}$$

$$\int_X g \circ f_n \, d\nu \to E(g(Y)) \text{ for all } g \in C([-\infty, \infty]).$$

Let

$$\mathcal{P}_m = \{\nu : \ \nu \text{ probability measure on } \mathcal{B}, \ \nu \ll m\}.$$

We define

$$\underline{f_n \xrightarrow{\mathcal{L}} Y} :\Longleftrightarrow f_n \xrightarrow{\nu} Y \ \text{ for all } \ \nu \in \mathcal{P}_m.$$

This type of convergence is called *strong distributional convergence.*

If $c \in [-\infty, \infty]$,

$$f_n \xrightarrow{\mathcal{L}} c \iff f_n \longrightarrow c \ \text{ in measure,}$$

which we also denote by $f_n \xrightarrow{m} c$.

Compactness theorem ([Aa4], [Aa1]): Let T be a non-singular ergodic transformation on the σ-finite measure space $(X, \mathcal{B}, m)$, and let $f_n : X \to \mathbb{R}$ $(n \geq 1)$ be measurable, satisfying

$$f_n \circ T - f_n \xrightarrow{m} 0 \quad \text{or} \quad \frac{f_n \circ T}{f_n} \xrightarrow{m} 1.$$

Then, for each subsequence (n_k) of $\mathbb{N}$ there exists a subsequence (m_ℓ) and a random variable Y on $[-\infty, \infty]$ such that

$$f_{m_\ell} \xrightarrow{\mathcal{L}} Y.$$

(The proof uses the Banach–Alaoglu theorem and the Riesz representation theorem. Needless to say, ergodicity is a central argument.) □

Corollary: Let $(X, \mathcal{B}, m, T)$ and (f_n) be as above, and let Y be a random variable on $[-\infty, \infty]$. Then

$$f_n \xrightarrow{\nu} Y \ \text{ for some } \nu \in \mathcal{P}_m \iff f_n \xrightarrow{\mathcal{L}} Y. \qquad \square$$

Application to Birkhoff sums

In the next section we shall apply the corollary in the following way.

Let T be conservative, ergodic and measure preserving on the σ-finite measure space $(X, \mathcal{B}, \mu)$. Suppose there exist $A \in \mathcal{B}$, $0 < \mu(A) < \infty$, $(a_n) \subseteq \mathbb{R}^+$ with $\lim\limits_{n\to\infty} a_n = \infty$, $\nu \in \mathcal{P}_\mu$, and a random variable Y on $[0, \infty]$ such that

$$\frac{1}{a_n} \sum_{k=0}^{n-1} 1_A \circ T^k \xrightarrow{\nu} \mu(A)\, Y\,.$$

Then,

$$\frac{1}{a_n}\sum_{k=0}^{n-1} f\circ T^k \xrightarrow{\mathcal{L}} \left(\int f\,d\mu\right) Y \quad \text{for all } f\in L_1(\mu)_+\,.$$

Proof: Let $f_n = \frac{1}{a_n}\sum_{k=0}^{n-1} 1_A\circ T^k$, $n\geq 1$. As $a_n\to\infty$, $f_n\circ T - f_n \xrightarrow{\mu} 0$ is obviously fulfilled. Thus, by the Corollary, $f_n \xrightarrow{\mathcal{L}} \mu(A)\,Y$, and Hopf's ergodic theorem then yields the full assertion. □

(3.2) The Darling–Kac theorem

Throughout, let T be conservative, ergodic and measure preserving on the σ-finite measure space $(X,\mathcal{B},\mu)$ where $\mu(X)=\infty$. We ask for the asymptotic distributional behaviour of the sojourn times

$$S_n 1_A := \sum_{k=0}^{n-1} 1_A\circ T^k \quad (n\geq 1),\;\; A\in\mathcal{B},\; 0<\mu(A)<\infty,$$

or, more generally, of

$$S_n f := \sum_{k=0}^{n-1} f\circ T^k \quad (n\geq 1),\;\; f\in L_1(\mu)_+\,.$$

Regarding absolutely normalized pointwise convergence we quote

Theorem 2.4.2 in [Aa1]: Given a sequence (a_n) of positive numbers, either

$$\liminf_{n\to\infty}\frac{1}{a_n}S_n f = 0 \quad \text{a.e.} \quad \text{for all } f\in L_1(\mu)_+\,,$$

or there exists a subsequence (n_k) of $\mathbb{N}$ such that

$$\lim_{k\to\infty}\frac{1}{a_{n_k}}S_{n_k} f = \infty \quad \text{a.e.} \quad \text{for all } f\in L_1(\mu)_+\,.$$

The Darling–Kac theorem, to be formulated below, shows that we have absolutely normalized convergence in weaker senses.

The limiting distributions occurring in this theorem are the so called Mittag–Leffler distributions (see e.g. [Fe2]).

Let $\alpha \in [0,1]$. The random variable $Y_\alpha > 0$ has the *normalized Mittag–Leffler distribution of order* α, if

$$E(Y_\alpha^p) = p!\,\frac{(\Gamma(1+\alpha))^p}{\Gamma(1+p\alpha)}, \quad p \in \mathbb{N}_0 .$$

- $E(Y_\alpha) = 1, \;\; \alpha \in [0,1]$ (whence 'normalized')
- $Y_1 = 1$ (borderline to finite ergodic theory)
- If $0 \le \alpha < 1$, the distribution of Y_α is absolutely continuous, in particular, $Y_0, Y_{1/2}$ have the densities

  $$f_{Y_0}(y) = e^{-y}, \quad f_{Y_{1/2}}(y) = \frac{2}{\pi}\, e^{-\frac{y^2}{\pi}} \quad (y \ge 0).$$

  For $0 < \alpha < 1$,

  $$f_{Y_\alpha}(y) = \frac{1}{\pi\alpha\Gamma(1+\alpha)} \sum_{k=1}^{\infty} \frac{\Gamma(1+k\alpha)}{k!} (\sin \pi k\alpha) \left(\frac{-y}{\Gamma(1+\alpha)}\right)^{k-1}, \quad y \ge 0.$$

- Connection with stable distributions

  For $0 < \alpha < 1$ let $X_\alpha > 0$ have the stable law of order α, specified by $E(e^{-tX_\alpha}) = e^{-t^\alpha}$, $t \ge 0$. Then,

  $$Y_\alpha \stackrel{\text{dist}}{=} \Gamma(1+\alpha) \cdot X_\alpha^{-\alpha} .$$

In our context, a common method to prove distributional limit theorems is the method of moments, based on Karamata's Tauberian theorem (KTT; see (2.4) of these notes). The following definition aims at this procedure.

Let $A \in \mathcal{B}$, $0 < \mu(A) < \infty$, and let

$$U_A(s) = \sum_{n=0}^{\infty} \frac{\mu(A \cap T^{-n}A)}{\mu(A)}\, e^{-ns}, \quad s > 0.$$

A is called a *moment set for* T if

$$\sum_{n=0}^{\infty} \left(\int_A (S_n 1_A)^p \, d\mu \right) e^{-ns} \sim p!\; \mu(A)\, \frac{(U_A(s))^p}{s} \quad (s \to 0)$$

for all $p \in \mathbb{N}_0$.

Note that $\int_A (S_n 1_A)^p \, d\mu / \mu(A)$ is the p-th moment of $S_n 1_A$ with respect to the initial distribution with density $\frac{1}{\mu(A)} 1_A$.

Darling–Kac Theorem ([DK], [Aa1], [Aa3], [Aa4]): Let T have a moment set A such that

$$U_A(s) \sim \left(\frac{1}{s}\right)^{\alpha} L\left(\frac{1}{s}\right) \quad (s \to 0)$$

with $\alpha \in [0,1]$ and L slowly varying at ∞. Then,

$$\frac{1}{a_n} S_n f \xrightarrow{\mathcal{L}} \left(\int f\, d\mu\right) Y_\alpha \text{ for all } f \in L_1(\mu)_+ ,$$

where

$$a_n = \sum_{k=0}^{n-1} \frac{\mu(A \cap T^{-k}A)}{\mu(A)^2}, \quad n \geq 1.$$

Proof: For all $p \geq 0$ we have

$$\sum_{n=0}^{\infty} \left(\int_A (S_n 1_A)^p \, d\mu\right) e^{-ns} \sim p!\, \mu(A) \left(\frac{1}{s}\right)^{1+p\alpha} \left(L\left(\frac{1}{s}\right)\right)^p \quad (s \to 0).$$

Since the sequences $\left(\int_A (S_n 1_A)^p \, d\mu\right)$ are non-decreasing, KTT yields

$$\int_A (S_n 1_A)^p \, d\mu \sim p!\, \mu(A)(n^\alpha L(n))^p / \Gamma(1+p\alpha) \quad (n \to \infty).$$

If we put $p = 1$, we see that

$$\mu(A)\, a_n \sim n^\alpha L(n)/\Gamma(1+\alpha) \quad (n \to \infty).$$

Therefore,

$$\int_A (S_n 1_A)^p \, d\mu \ \sim\ p!\, (\mu(A))^{p+1} \frac{(\Gamma(1+\alpha))^p}{\Gamma(1+p\alpha)}\, a_n^p \quad (n \to \infty)$$

for all $p \geq 0$, i.e.,

$$\frac{1}{a_n} S_n 1_A \xrightarrow{\nu} \mu(A)\, Y_\alpha ,$$

where ν is the probability measure with density $\frac{1}{\mu(A)} 1_A$. In view of (3.1) the proof is complete. □

Regarding sufficient conditions for the existence of moment sets, we quote

Theorem 3.7.2 in [Aa1]: Let T be pointwise dual ergodic with return sequence (a_n). Then, every set $A \in \mathcal{B}$, $0 < \mu(A) < \infty$, satisfying

$$(\star) \qquad \sup_{n \geq 1} \left\| \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A \right\|_{L_\infty(A)} < \infty$$

is a moment set for T.

Corollary: If T is pointwise dual ergodic, and the return sequence (a_n) is regularly varying with index α, then

$$\frac{1}{a_n} S_n f \xrightarrow{\mathcal{L}} \left(\int f \, d\mu \right) Y_\alpha \quad \text{for all } f \in L_1(\mu)_+ . \qquad \square$$

Evidently, $(\star)$ is satisfied if A is a Darling–Kac set for T, i.e., if

$$\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A \longrightarrow \mu(A) \quad \text{almost uniformly on } A.$$

We shall content ourselves with a proof of

$(\star\star)$: Darling–Kac sets are moment sets.

Before proving this let us return once more to our

Examples: Let $T : [0,1] \to [0,1]$ satisfy (T1) - (T5), and let

$$T(x) = x \pm a_j |x - x_j|^{p_j+1} + o(|x - x_j|^{p_j+1}) \quad (x \to x_j)$$

with $a_j > 0$, $p_j \geq 1$ $(j \in J)$, and $p = \max\{p_j : j \in J\}$. Then, for some constant $c = c(T)$,

$$\text{if } p = 1, \quad \frac{\log n}{n} S_n f \xrightarrow{\mathcal{L}} c \int f \, d\mu, \quad \text{and,}$$

$$\text{if } p > 1, \quad \frac{1}{n^{1/p}} S_n f \xrightarrow{\mathcal{L}} c \left(\int f \, d\mu \right) Y_{1/p}$$

for all $f \in L_1(\mu)_+$.

Proof of $(\star\star)$: Let $S_n := \sum_{k=1}^{n} 1_A \circ T^k$, $n \geq 0$. For each $p \geq 1$ there exist numbers $c_p(j)$, $1 \leq j \leq p$, with $c_p(p) = p!$ such that

$$S_n^p = \sum_{j=1}^{p} c_p(j) \, S_n^{(j)} \quad \text{where} \quad S_n^{(j)} = \binom{S_n}{j} \quad (n, j \geq 0).$$

As $U_A(s) \to \infty$ $(s \to 0)$ it therefore suffices to prove

$$M_p(s) := \sum_{n=0}^{\infty} \left(\int_A S_n^{(p)} \, d\mu \right) e^{-ns} \sim \mu(A) \, \frac{(U_A(s))^p}{s} \quad (s \to 0).$$

We proceed by induction, based on the recursion formula

$$S_n^{(p+1)} = \sum_{k=1}^{n} \left(1_A \cdot S_{n-k}^{(p)}\right) \circ T^k \quad (p \geq 0),$$

which is left as an exercise. We have

$$\int_A S_n^{(p+1)}\, d\mu = \int_A \sum_{k=1}^{n} \hat{T}^k 1_A \cdot S_{n-k}^{(p)}\, d\mu, \quad \text{and thus}$$

$$M_{p+1}(s) = \int_A \left(\sum_{n=1}^{\infty} \hat{T}^n 1_A\, e^{-ns}\right) \left(\sum_{n=0}^{\infty} S_n^{(p)} e^{-ns}\right) d\mu \quad (s > 0).$$

In the proof of the asymptotic renewal equation ((2.4) of these notes) we had

$$\sum_{n=1}^{\infty} \hat{T}^n 1_A\, e^{-ns} \sim U_A(s) \quad (s \to 0) \quad \text{almost uniformly on } A.$$

Therefore, $M_{p+1}(s) \sim U_A(s) \cdot M_p(s)$ $(s \to 0)$ for all $p \geq 0$. Taking into account that $M_0(s) \sim \dfrac{\mu(A)}{s}$ $(s \to 0)$, we get the desired relation.

□

(3.3) The Dynkin–Lamperti arc-sine laws

In their most elementary form, arc-sine laws occur in the study of the simple symmetric random walk on the integers. A detailed presentation for this case, including many comments, is Chap. III in [Fel], which in fact motivated the considerations of this section.

Roughly, there are two main types of arc-sine laws:

- *the arc-sine laws of renewal theory* (Dynkin–Lamperti arc-sine laws for last visits and related variables)
- *the arc-sine laws for random walks* (Lévy–Spitzer arc-sine laws for sojourn times, positions of maxima, ...).

Theorems 1 and 2 in Chap. III of [Fel] introduce both types in terms of the simple symmetric random walk on $\mathbb{Z}$ starting from the origin. We quote them in their limiting form.

1. Let Z_n denote the time of the last visit to the origin in the time interval $[0, n]$, $n \geq 1$. Then

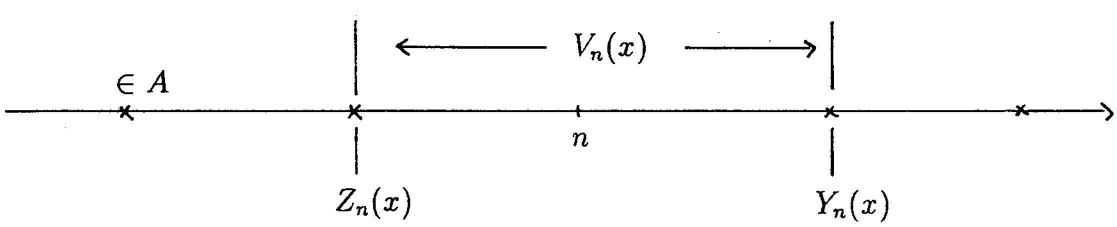
$$(\star) \qquad \text{Prob}\left(\frac{1}{n} Z_n \leq x\right) \to \frac{2}{\pi} \arcsin \sqrt{x}, \quad 0 \leq x \leq 1.$$

2. Let N_n be the number of times in the inverval $[0, n]$ the "moving particle" is on the positive side. Then $(\star)$ holds for the sequence (N_n) as well.

Here we consider the first type of arc-sine laws for pointwise dual ergodic transformations. (For specific one-dimensional maps results of the second type have recently been obtained in [Th7].)

Throughout, let T be a conservative, ergodic, measure preserving transformation on the σ-finite measure space $(X, \mathcal{B}, \mu)$, and let $A \in \mathcal{B}$ have finite positive measure. Given $n \geq 1$, we consider the following variables:

- $Z_n(x) = \max\{k \leq n : T^k(x) \in A\}$, $x \in \bigcup_{k=0}^{n} T^{-k}A =: A_n$,
  $Z_n(x) = 0$, $x \notin A_n$,
- $Y_n(x) = \inf\{k > n : T^k(x) \in A\}$, $x \in X$, and
- $V_n = Y_n - Z_n$:



In renewal theoretic language, $n - Z_n$ is the spent waiting time, $Y_n - n$ the residual waiting time, and V_n the total waiting time if the process is inspected at time n. As T is conservative and ergodic,

$$A_n \uparrow X \quad (\text{mod } \mu) \quad \text{and} \quad Y_n < \infty \quad \text{a.e. for all } n \geq 1.$$

The asymptotic distributional behaviour of these functions, considered as random variables on $(X, \mathcal{B}, \nu)$ with $\nu \in \mathcal{P}_\mu$, can be illustrated nicely by means of computer experiments with maps from our class.

The limiting distributions

For $0 < \alpha < 1$ let ζ_α be a random variable on $(0,1)$ with density

$$f_{\zeta_\alpha}(x) = \frac{\sin \pi\alpha}{\pi} \frac{1}{x^{1-\alpha}(1-x)^\alpha}, \quad 0 < x < 1,$$

i.e., $\zeta_\alpha \sim B(\alpha, 1-\alpha)$ [Fig.(2) on p. 13], and let η_α be a random variable on $(0,\infty)$ with density

$$f_{\eta_\alpha}(x) = \frac{\sin \pi\alpha}{\pi} \frac{1 - (\max\{1-x, 0\})^\alpha}{x^{1+\alpha}}, \quad x > 0 \quad \text{[Fig.(4) on p. 13]}.$$

The density of $1/\zeta_\alpha$ is given by

$$f_{1/\zeta_\alpha}(x) = \frac{\sin \pi\alpha}{\pi} \frac{1}{x(x-1)^\alpha}, \quad x > 1 \quad \text{[Fig.(3) on p. 13]}.$$

Extending these families of distributions continuously to the parameter interval $[0,1]$ yields

$$\zeta_0 = 0 \quad (1/\zeta_0 = \infty), \quad \zeta_1 = 1, \quad \text{and} \quad \eta_0 = \infty, \quad \eta_1 = 0.$$

Remarks:

- To verify that $f_{\zeta_\alpha}, f_{\eta_\alpha}$ are probability densities note that

$$\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}.$$

- The moments of ζ_α are given by

$$E\left(\zeta_\alpha^p\right) = (-1)^p \binom{-\alpha}{p}, \quad p \geq 0; \text{ in particular, } E(\zeta_\alpha) = \alpha.$$

- For η_α the parameter can be recovered from $\text{Prob}(\eta_\alpha > 1) = \tau(\alpha)$, where $\tau(\alpha) = \dfrac{\sin \pi\alpha}{\pi\alpha}$, $0 < \alpha \leq 1$, $\tau(0) = 1$.

Dynkin–Lamperti Theorem ([Dy], [La], [Th6]): Let T be pointwise dual ergodic with return sequence (a_n), and let $A \in \mathcal{B}$, $0 < \mu(A) < \infty$, be a uniform set for T, i.e., there exists a probability density f such that

$$\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f \;\longrightarrow\; 1 \quad \text{almost uniformly on } A.$$

Then, for $\alpha \in [0,1]$, the following assertions are equivalent:

(1) (a_n) is regularly varying with index α

(2) $\frac{1}{n} Z_n \xrightarrow{\mathcal{L}} \zeta_\alpha$

(3) $\frac{1}{n} Y_n \xrightarrow{\mathcal{L}} 1/\zeta_\alpha$

(4) $\frac{1}{n} V_n \xrightarrow{\mathcal{L}} \eta_\alpha$

(2)' $\frac{1}{n} Z_n \xrightarrow{\nu} \zeta$ for some $\nu \in \mathcal{P}_\mu$ and some random variable ζ on $[0,1]$ with $E(\zeta) = \alpha$

(3)', (4)': analogous to (2)'.

By Theorem 2 in (2.4) of these notes a further equivalent condition is that the wandering rate $(w_n(A))$ varies regularly with exponent $1-\alpha$. □

Examples: Let $T : [0,1] \to [0,1]$ satisfy (T1) - (T5), and let

$$T(x) = x \pm a_j |x - x_j|^{p_j+1} + o(|x - x_j|^{p_j+1}) \quad (x \to x_j)$$

with $a_j > 0$, $p_j \geq 1$ $(j \in J)$, and $p = \max\{p_j \in J\}$. Then the above statements hold with $\alpha = 1/p$ for all sets $A \in \mathcal{B}$ of positive measure which are bounded away from the indifferent fixed points.

The rest of this section sketches part of the proof of the Theorem. First we settle the question of dependence on the initial distribution.

Lemma: If (f_n) denotes any of the sequences $\left(\frac{1}{n} Z_n\right)$, $\left(\frac{1}{n} Y_n\right)$ or $\left(\frac{1}{n} V_n\right)$, then

$$f_n \circ T - f_n \xrightarrow{\mu} 0.$$

□

Thus the Corollary in (3.1) shows that it suffices to prove the equivalences asserted in the Theorem for the probability measure ν with density f.

Moreover, some of the implications are immediate from

$$\{Z_n \leq k,\ Y_n > m\} = \{Z_m \leq k\}, \quad 1 \leq k \leq n \leq m.$$

The main steps left are:

(i) (1) $\Longrightarrow \frac{1}{n} Z_n \xrightarrow{\ \nu\ } \zeta_\alpha$, and

(ii) $\frac{1}{n} V_n \xrightarrow{\ \nu\ } \eta$ for some random variable on $[0, \infty]$ with
Prob$(\eta > 1) = \tau(\alpha) \Longrightarrow$ (1).

We only sketch the proof of (i).

Again we use the method of moments, based on KTT. The key tool is a generalized version of the asymptotic renewal equation. Let U, Q be defined as in (2.4) of these notes:

$$U(s) = \sum_{n=0}^{\infty} \nu(T^{-n}A)\, e^{-ns}, \quad Q(s) = \sum_{n=0}^{\infty} \frac{\mu(A \cap \{\varphi > n\})}{\mu(A)} e^{-ns} \quad (s > 0),$$

where φ is the first return time of A, and let $M_p(s)$ $(p \geq 0)$ be the Laplace transforms of the p-th moments of (Z_n) with respect to ν:

$$M_p(s) = \sum_{n=1}^{\infty} \left(\int_{A_n} Z_n^p \, d\nu \right) e^{-ns} \quad (s > 0).$$

We prove:

$$(\star) \qquad M_p(s) \sim p!\ E(\zeta_\alpha^p) \left(\frac{1}{s} \right)^{p+1} \quad (s \to 0), \quad p \geq 0.$$

Since the sequences $\left(\int_{A_n} Z_n^p \, d\nu \right)$ are non-decreasing, KTT then yields

$$\int_{A_n} \left(\frac{1}{n} Z_n \right)^p d\nu \ \longrightarrow\ E(\zeta_\alpha^p) \quad \text{for all } p \geq 0,$$

i.e., $\frac{1}{n} Z_n \xrightarrow{\ \nu\ } \zeta_\alpha$. We consider, in fact, the transforms

$$\tilde{M}_p(s) = \sum_{n=1}^{\infty} \left(\int_{A_n} (n - Z_n)^p \, d\nu \right) e^{-ns} \quad (s > 0,\ p \geq 0),$$

for which we have the

Generalized asymptotic renewal equation: For all $p \geq 0$,

$$\tilde{M}_p(s) \sim (-1)^p U(s) \cdot Q^{(p)}(s) \sim \frac{1}{s} \frac{(-1)^p Q^{(p)}(s)}{Q(s)} \quad (s \to 0).$$

(For $p = 0$, the first equivalence gives $U(s) \cdot Q(s) \sim \frac{1}{s} \quad (s \to 0)$.)

The proof of the general case uses similar calculations and arguments as the proof for $p = 0$. $\square$

Now assume (a_n) is regularly varying with index α. Then $p = 0$ yields

$$Q(s) \sim \left(\frac{1}{s}\right)^{1-\alpha} L\left(\frac{1}{s}\right) \quad (s \to 0)$$

with L slowly varying at ∞. By the Monotone Density Theorem of regular variation theory (see e.g. [BGT]) it follows that

$$Q^{(p)}(s) \sim p! \binom{\alpha - 1}{p} \left(\frac{1}{s}\right)^{p+1-\alpha} L\left(\frac{1}{s}\right) \quad (s \to 0)$$

('differentiation' by treating L as a constant). Therefore the generalized asymptotic renewal equation gives

$$\tilde{M}_p(s) \sim p! \ E(\zeta_{1-\alpha}^p) \left(\frac{1}{s}\right)^{p+1} \quad (s \to 0), \ \ p \geq 0.$$

Since $\zeta_{1-\alpha} \stackrel{\text{dist}}{=} 1 - \zeta_\alpha$ it is not difficult to verify that this implies $(\star)$. $\square$

We mention that our procedure of generalizing the renewal equation also slightly simplifies the usual proof for the classical case.

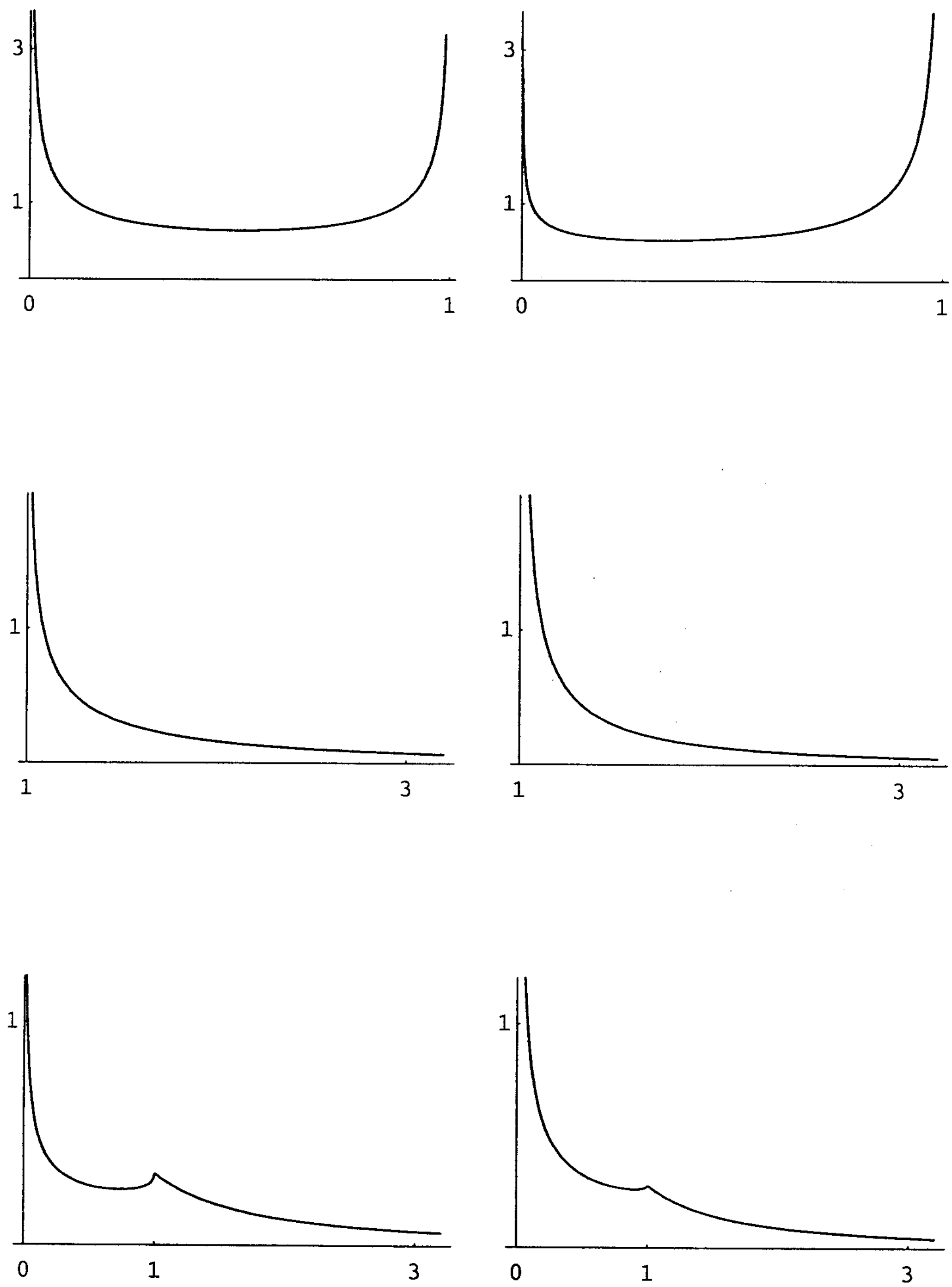
Limiting densities in the Dynkin–Lamperti theorem

$\alpha = 1/2$ $\alpha = 2/3$

(2)

(3)

(4)



References

[Aa1] J. Aaronson, *An introduction to infinite ergodic theory,* Amer. Math. Soc., 1997

[Aa3] J. Aaronson, *Random f-expansions,* Ann. Probab. **14** (1986), 1037 - 1057

[Aa4] J. Aaronson, *The asymptotic distributional behaviour of transformations preserving infinite measures,* J. Anal. Math. **39** (1981), 203-234

[BGT] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation.* Cambridge University Press, Cambridge, 1987

[DK] D. A. Darling and M. Kac, *On occupation times for Markoff processes,* Trans. Amer. Math. Soc. **84** (1957), 444 - 458

[Dy] E. B. Dynkin, *Some limit theorems for sums of independent random variables with infinite mathematical expectations,* Selected Transl. in Math. Statist. and Probability **1** (1961), 171 - 189

[Fe1] W. Feller, *An introduction to probability theory and its applications, Vol. I.* John Wiley & Sons, New York, 1968

[Fe2] W. Feller, *An introduction to probability theory and its applications, Vol. II.* John Wiley & Sons, New York, 1971

[La] J. Lamperti, *Some limit theorems for stochastic processes,* J. Math. Mech. **7** (1958), 433 - 448

[Th6] M. Thaler, *The Dynkin–Lamperti arc-sine laws for measure preserving transformations,* Trans. Amer. Math. Soc. **350** (1998), 4593 - 4607

[Th7] —, *A limit theorem for sojourn times near indifferent fixed points of one-dimensional maps,* Preprint, Salzburg, 2000

[Z4] R. Zweimüller, *Probabilistic properties of dynamical systems with infinite invariant measure,* Master's Thesis, Salzburg, 1995