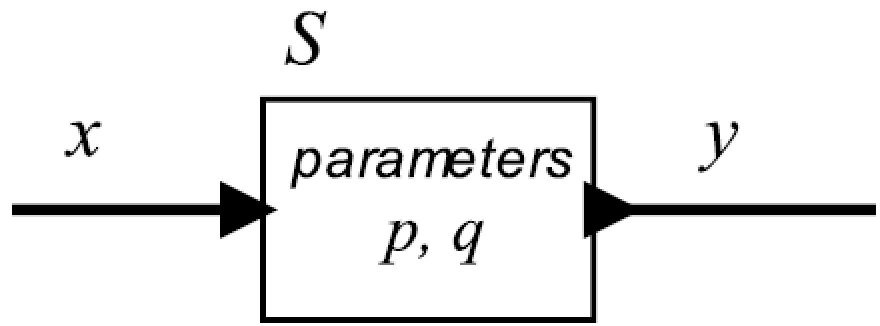
# Embedded and Cyber-Physical Systems

## - Modeling continuous behavior -

Reference book: Edward A. Lee and Pravin Varaiya, Structure and Interpretation of Signals and Systems, Second Edition, LeeVaraiya.org, ISBN 978-0-578-07719-2, 2011.

# Actor Model of Systems

- A *system* is a function that accepts an input *signal* and yields an output signal.




- The domain and range of the system function are sets of signals, which themselves are functions.

$$x\colon \mathbb{R} \to \mathbb{R}, \quad y\colon \mathbb{R} \to \mathbb{R}$$

$$S\colon X \to Y$$

$$X = Y = (\mathbb{R} \to \mathbb{R})$$

- Parameters may affect the definition of the function *S*.

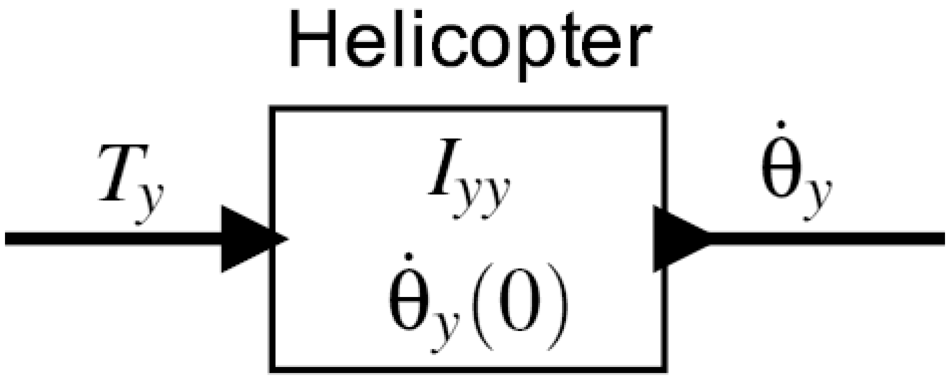
# Helicopter example contd.

A helicopter without a tail rotor, like the one below, will spin uncontrollably

Control system problem:
Apply torque using the tail rotor

# Actor model of the helicopter

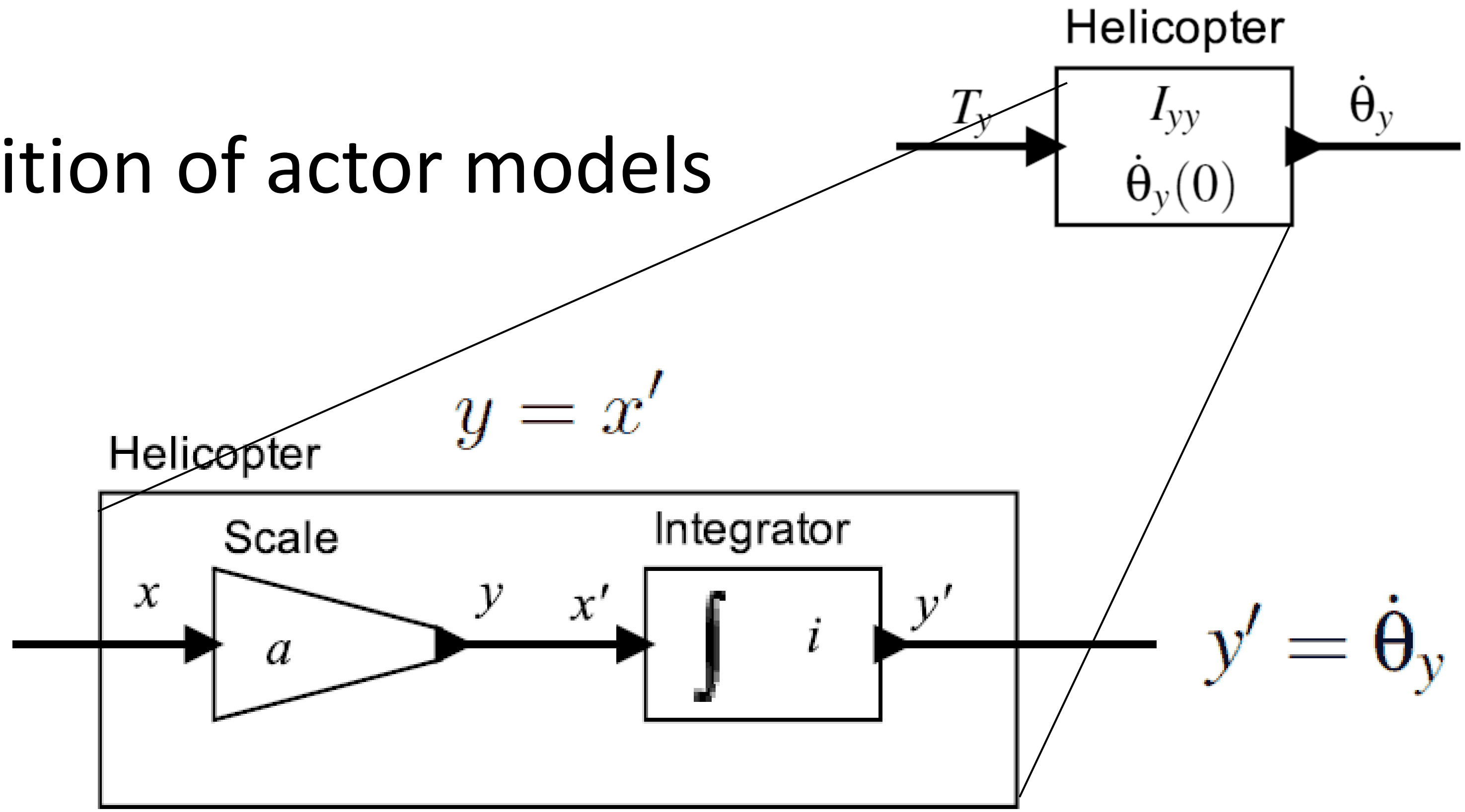
- Input is the net torque of the tail rotor. Output is the angular velocity around the *y* axis.




Parameters of the model are shown in the box. The input and output relation is given by the equation to the right.

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

# Composition of actor models




$$x = T_y \qquad y' = \dot{\theta}_y$$

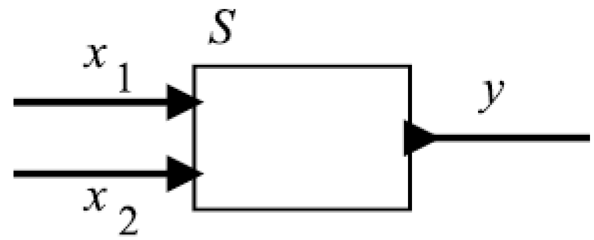
$$\forall\, t \in \mathbb{R}, \quad y(t) = ax(t) \qquad y'(t) = i + \int_0^t x'(\tau)d\tau$$

$$y = ax$$

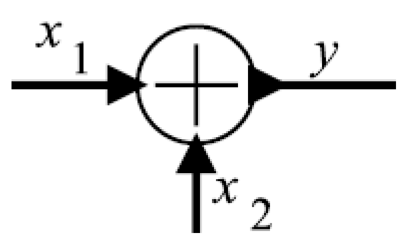
$$a = 1/I_{yy}$$

$$i = \dot{\theta}_y(0)$$

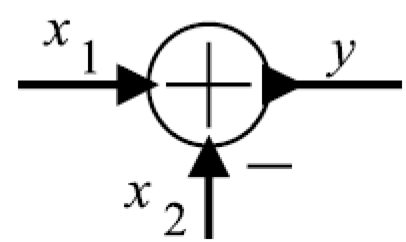
# Actor models with multiple inputs




$$S\colon (\mathbb{R} \to \mathbb{R})^2 \to (\mathbb{R} \to \mathbb{R})$$

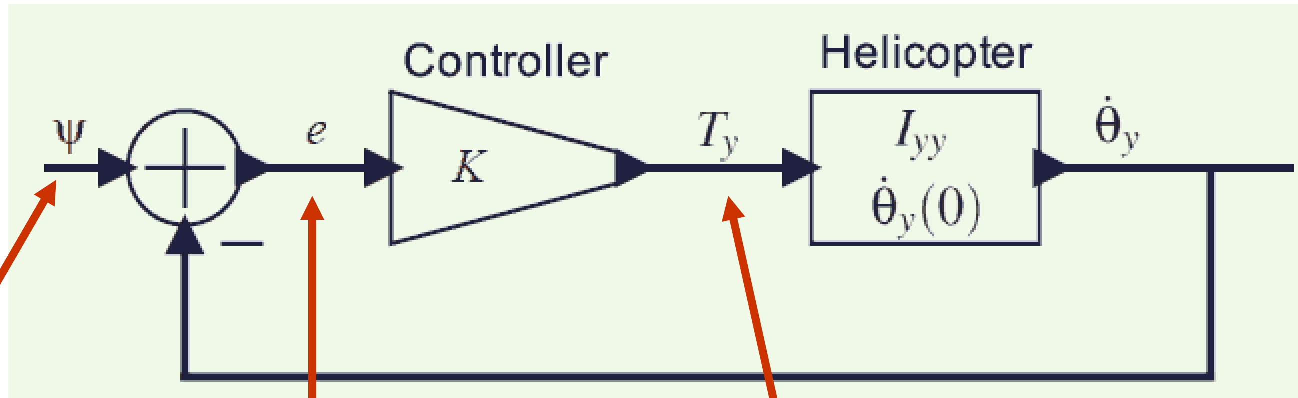



$$\forall\, t \in \mathbb{R}, \quad y(t) = x_1(t) + x_2(t)$$




$$(S(x_1, x_2))(t) = y(t) = x_1(t) - x_2(t)$$

# Proportional controller




desired angular velocity

error signal

net torque

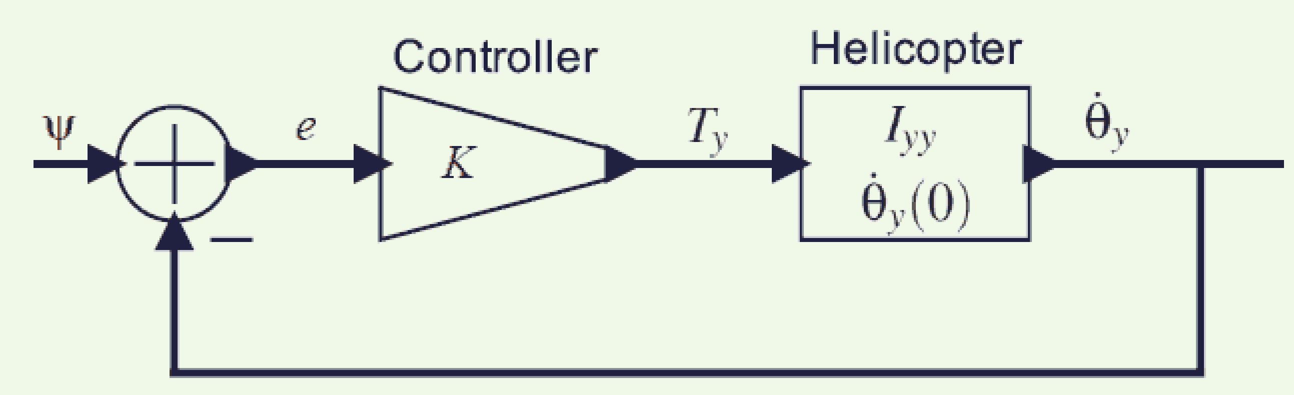
$$e(t) = \psi(t) - \dot{\theta}_y(t)$$

$$T_y(t) = Ke(t)$$

$$\begin{aligned}\dot{\theta}_y(t) &= \dot{\theta}_y(0) + \frac{1}{I_{yy}}\int_0^t T_y(\tau)d\tau \\ &= \dot{\theta}_y(0) + \frac{K}{I_{yy}}\int_0^t (\psi(\tau) - \dot{\theta}_y(\tau))d\tau\end{aligned}$$

Note that the angular velocity appears on both sides, so this equation is not trivial to solve.

# Feedback Control




$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t (\psi(\tau) - \dot{\theta}_y(\tau))d\tau$$

Assume that helicopter is initially at rest,

$$\dot{\theta}(0) = 0,$$

and that the desired signal is

$$\psi(t) = au(t) \qquad u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

for some constant $a$.

Let‘s see how to derive the solution!

# Convolution function

- Definition: $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$

- Properties (on a need-to-know basis):

a) $f * g = g * f$

b) $(f * g) * h = f * (g * h)$

c) $\left. \begin{array}{l} f(t) = e^{\lambda t} u(t) \\ g(t) = u(t) \end{array} \right\} \rightarrow (f * g)(t) = \frac{1 - e^{\lambda t}}{-\lambda} u(t)$

# Laplace transform

$f(t)$ is **absolutely integrable** if $\int_{-\infty}^{\infty} |f(t)| dt$ exists and is finite

- Laplace transform: $\hat{F}(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$

  where $s \in \{\sigma + i\omega \in \mathbb{C} \mid f(t)e^{-\sigma t} \text{ is absolutely integrable}\}$

- Properties:

$$t^n \rightarrow \frac{n!}{s^{n+1}}$$

$$\int f(t) \rightarrow \frac{\hat{F}(s)}{s}$$

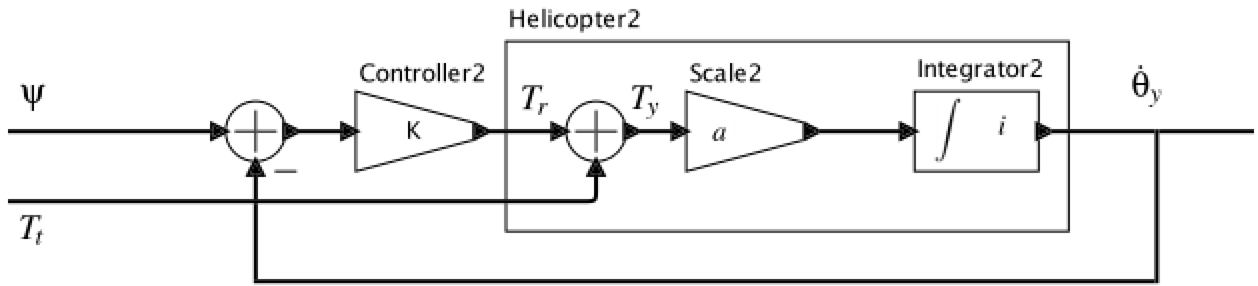
$$u(t) \rightarrow \frac{1}{s}$$

$$e^{-\lambda t} \cdot u(t) \rightarrow \frac{1}{s+\lambda}$$

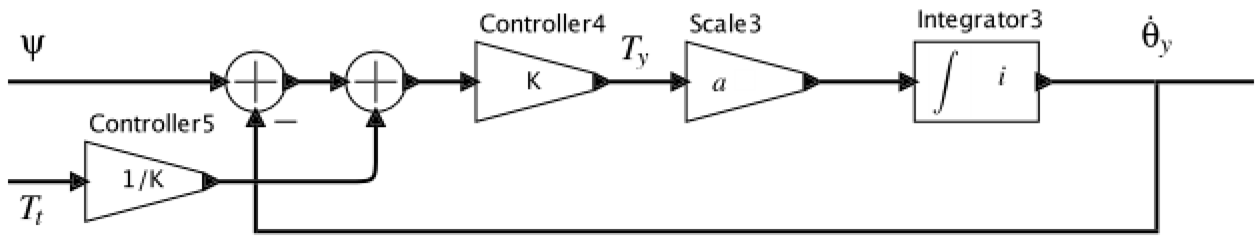
$$(f * g)(t) \rightarrow \hat{F}(s) \cdot \hat{G}(s)$$

# Your solution here

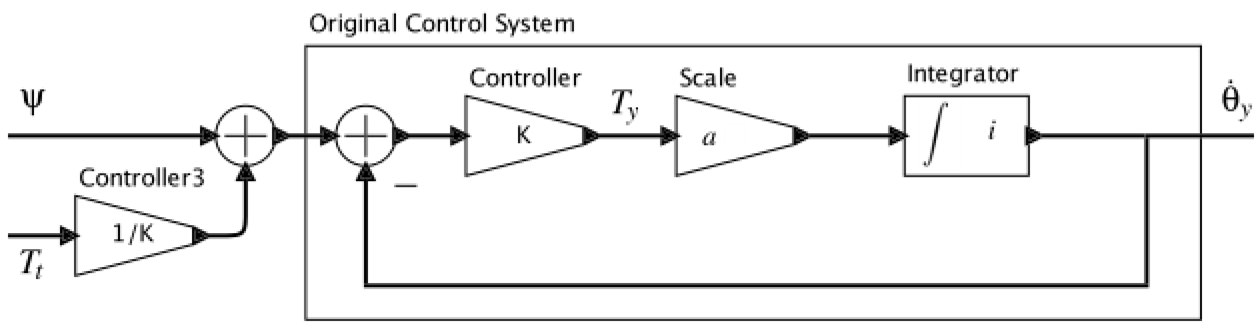
# How about the main rotor?




(a)




(b)




(c)

# Solution with top rotor included

- Suppose that the torque given by the top rotor is

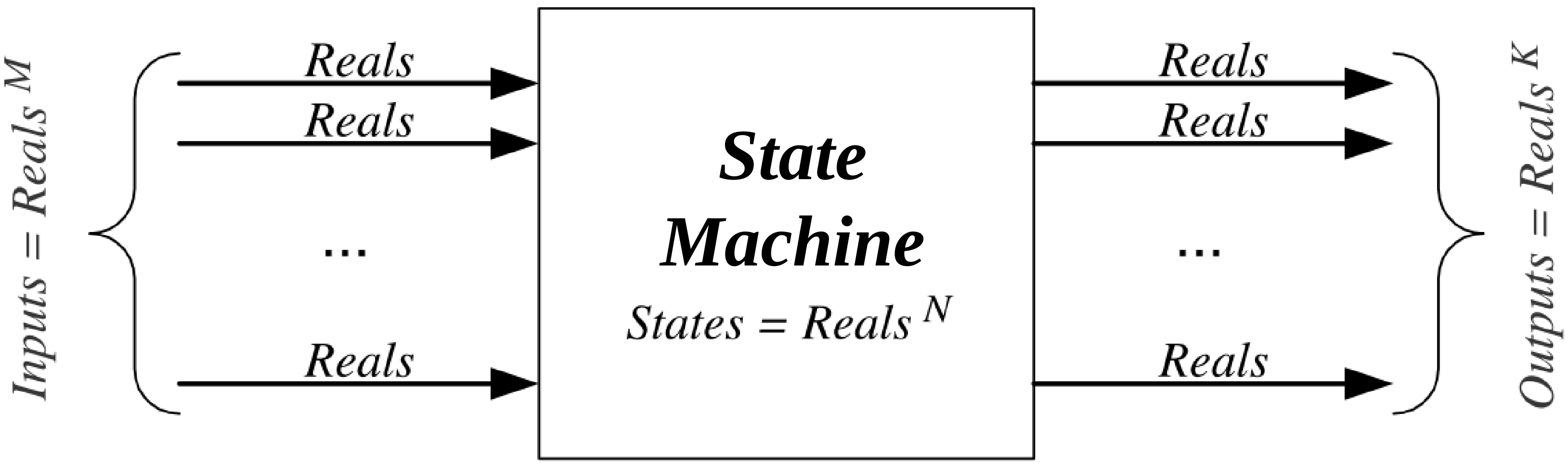
$$T_t = bu(t)$$

- The desired angular rotation is zero
- The solution is then

$$\dot{\theta}_y(t) = \frac{b}{K} u(t)(1 - e^{-Kt/I_{yy}})$$

- Does the controller achieve its objective?

# Discrete model of systems




$$StateMachine = (States, Inputs, Outputs, update, initialState)$$

$s(n)$ $x(n)$ $y(n)$ $s(0)$

$$\begin{aligned} s(0) &= initialState, \\ \forall n \geq 0,\ (s(n+1), y(n)) &= update(s(n), x(n)) \end{aligned}$$

# The discrete state space model

$$\begin{aligned} s(n+1) &= nextState(s(n), x(n)), \\ y(n) &= output(s(n), x(n)). \end{aligned}$$

- Linear system:
  - Initial state is zero
  - The *output* and *nextState* functions are linear
- Time-invariant: the two functions don‘t change with time
- LTI: linear time-invariant

# State space model of LTI systems

- The functions are represented by matrices

$$
\begin{aligned}
s(n+1) &= As(n) + Bx(n), \\
y(n) &= Cs(n) + Dx(n).
\end{aligned}
$$

# Continuous time state space models

For an LTI single-input, single-output system:

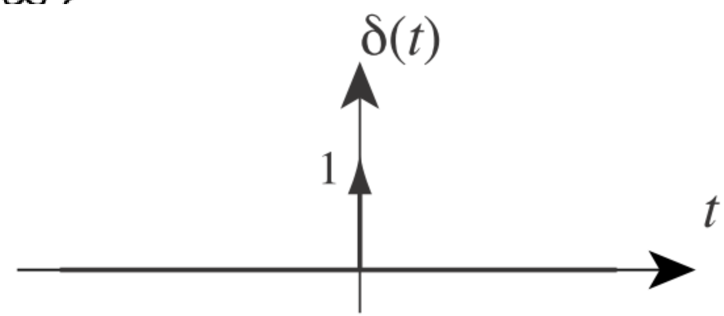
$$\forall\, t \in \mathbb{R}_+, \quad \dot{z}(t) = Az(t) + bv(t)$$

$$w(t) = c^T z(t) + dv(t)$$

- $z\colon \mathbb{R}_+ \to \mathbb{R}^N$ gives the state response;
- $\dot{z}(t)$ is the derivative with respect to time of $z$ evaluated at $t \in \mathbb{R}_+$;
- $v\colon \mathbb{R}_+ \to \mathbb{R}$ is the input signal; and
- $w\colon \mathbb{R}_+ \to \mathbb{R}$ is the output signal.

# The Dirac delta (generalized) function

$\delta \colon \mathbb{R} \to \mathbb{R}_{++} \qquad \mathbb{R}_{++} = \mathbb{R} \cup \{\infty, -\infty\}$




$\forall\, t \in \mathbb{R}$ where $t \neq 0, \quad \delta(t) = 0$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1 \quad \text{for any } \varepsilon > 0 \text{ in } \mathbb{R}_{++}$$

Given any signal $x \colon \mathbb{R} \to \mathbb{R}$,

$$\forall\, t \in \mathbb{R}, \quad x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

# Impulse response and I/O

- LTI system $S\colon [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$
- Impulse response: $\forall t \in \mathbb{R}, \quad h(t) = S(\delta)(t)$

let $x$ be any input signal

let $y = S(x)$, be the corresponding output signal

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

Since $S$ is LTI, the output is a sum (integral) of the responses to each of the components (the integrand for fixed $\tau$), or

$$\forall\, t \in \mathbb{R}, \quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = (x*h)(t) = (h*x)(t)$$

# The transfer function

$$x(t) \quad \rightarrow \quad \hat{X}(s)$$

$$y(t) \quad \rightarrow \quad \hat{Y}(s)$$

$$h(t) \quad \rightarrow \quad \hat{H}(s)$$

$$y(t) = (h * x)(t) \rightarrow \hat{Y}(S) = \hat{H}(s) \cdot \hat{X}(S)$$

The transfer function is the Laplace transform of the impulse response!

- Write the transfer function of the helicopter system.

# Stability

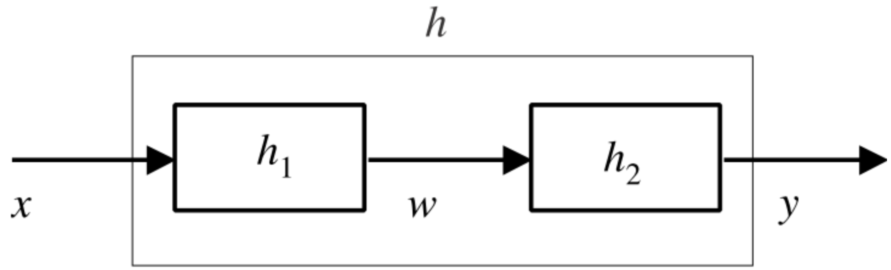
A system is said to be **bounded-input bounded-output stable** (**BIBO stable** or just **stable**) if the output signal is bounded for all input signals that are bounded.

- A continuous time LTI system is stable if and only if its impulse response is absolutely integrable
- For rational Laplace transforms:

$$\hat{X}(s) = \frac{A(s)}{B(s)}$$

- A continuous time causal system is stable if and only if all the roots of B(s) have negative real parts

# Model composition: Cascade

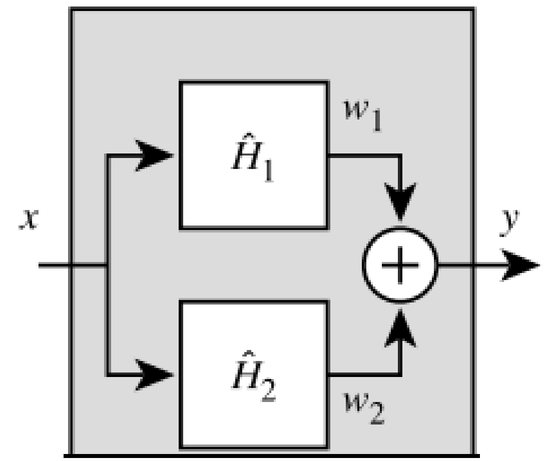



- Write y(t) as a function of x(t)

- What is the transfer function of the composition?

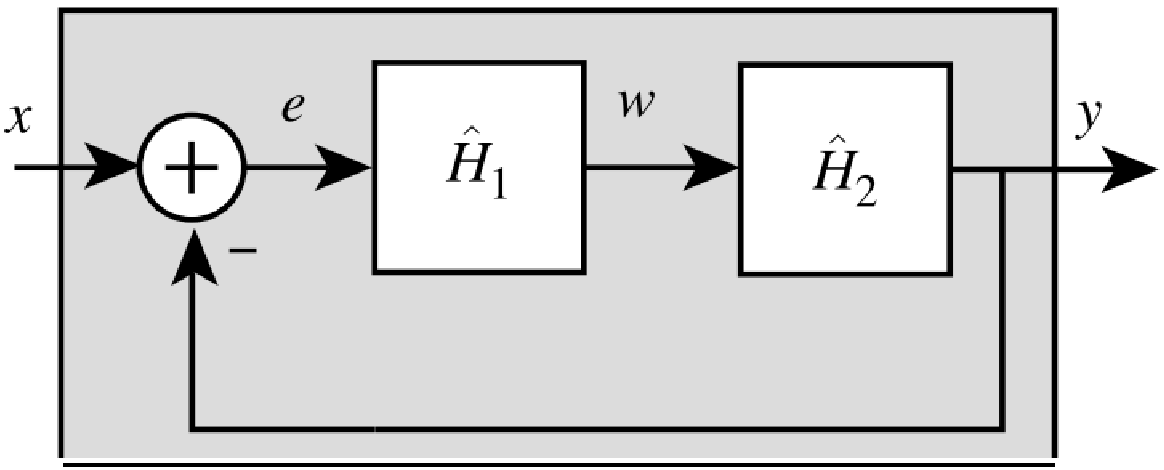
- Can we use this to stabilize a system?

# Model composition: Parallel




- Write y(t) as a function of x(t)
- What is the transfer function of the composition?
- Can we use this for stabilization?

# Model composition: Feedback




- What is the transfer function of the composition?

- Can we use this for stabilization?

# Helicopter questions

- What is the tf of the helicopter model?
- What is the tf of the proportional controller?
- What is the tf of the closed-loop systems?
- Are the three systems above stable?

- What other effect has the tail rotor torque?
- Think about how to counteract that effect
  - New control strategy, new actuator on the helicopter

# Food for thought: helicopter spinning

- Watch the two video examples
- The helicopters are similar in construction
- Why do they behave differently?