PARTIAL DIFFERENTIAL EQUATIONS

Lecture 1

1. Introduction

Def. An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0, \qquad x \in \Omega \subset \mathbb{R}^n, \quad -(*)$$

is a k-th order system of PDE, where

$$F : \mathbb{R}^{Nn^k} \times \mathbb{R}^{Nn^{k-1}} \times \dots \times \mathbb{R}^{Nn} \times \mathbb{R}^N \times \Omega \to \mathbb{R}^N$$

is given and

$$u = (u^1, \dots, u^N) : \Omega \to \mathbb{R}^N$$

is the unknown.

Remark. When $N = 1$ we say $(*)$ is a k-th order partial differential equation.

Notation. If $u = (u^1, \dots, u^N) : \Omega \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}$, is open, then $Du : \Omega \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ s.t. $Du(x) : \mathbb{R}^n \to \mathbb{R}^N$ is the linear map defined by

$$Du(x)(e_j) = \lim_{t \to 0} \frac{u(x + te_j) - u(x)}{t}.$$

Equivalently, we can write

$$Du(x)(e_j) = D_j u(x) = \partial_j u(x) = \frac{\partial u}{\partial x} = u_{x_j}.$$

Note that
$$\begin{cases} Du = (Du^1, \dots, Du^N) = \text{gradient of } u \\ D^2 u = (D^2_{ij} u) = \text{Hessian matrix of 2nd derivatives.} \end{cases}$$

If $\beta = (\beta_1, \dots, \beta_n)$ where $\beta_j = \text{integers} \geq 0$ is a multi-index we define

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} = D^{|\beta|} u(x)(e_{\beta_1}, \dots, e_{\beta_k})$$

where $|\beta| = \beta_1 + \dots + \beta_n$.

Note that for an open bounded $\Omega \subset \mathbb{R}^n$ we denote:

$C^k(\Omega)$ = the set of functions having derivatives of order $\leq k$ continuous in Ω.

$C^k(\bar{\Omega})$ = the set of functions in $C^k(\Omega)$ whose derivatives of order $\leq k$ have continuous extensions to $\bar{\Omega}$.

If $f \in C^k(\Omega)$ then the Taylor series is given by

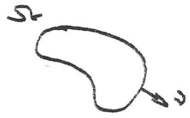
$$f(x+h) = f(x) + Df(x)(h) + \frac{1}{2} D^2 f(x)(h,h) + \dots + \frac{1}{(k-1)!} D^{k-1} f(x)(h, \dots, h) + \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} D^k f(x + th)(h, \dots, h)\, dt.$$

e.g. when $k = 2$ we have

$$f(x+h) = f(x) + Df(x)(h) + \int_0^1 (1-t) D^2 f(x + th)(h,h)\, dt.$$

Def. We say $\partial\Omega$ is C^k if for each $x_0 \in \partial\Omega$ we can locally represent the boundary as the graph of a C^k-function.

Def. If $\partial\Omega$ is C^1, then there exists an outward pointing unit normal $\underline{\nu} = (\nu_1, \dots, \nu_n)$ and if $u \in C^1(\bar{\Omega})$ we call



$$\frac{\partial u}{\partial \nu} = \underline{\nu} \cdot Du$$

the <u>outward normal derivative</u> of u.

<u>Gauss-Green Theorem.</u> If $u \in C^1(\bar{\Omega})$ then

$$\int_\Omega \frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} u\, \nu_i\, d\sigma(x).$$

<u>Corollary (integration by parts).</u>

$$\int_\Omega \frac{\partial u}{\partial x_i} v\, dx = -\int_\Omega u \frac{\partial v}{\partial x_i}\, dx + \int_{\partial\Omega} u v\, \nu_i\, d\sigma(x).$$

(i) Green's first identity

$$\boxed{\int_\Omega v \Delta u + \int_\Omega Du \cdot Dv = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, d\sigma}$$

(ii) Green's second identity

$$\boxed{\int_\Omega (u \Delta v - v \Delta u)\, dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}\right) d\sigma}$$

2. Classification

Def. (i) The PDE $(*)$ is said to be <u>linear</u> if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x)$$

for given a_α and f. It is said to be homogeneous if $f = 0$.

(ii) — " — <u>semilinear</u> if it is of the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^k u + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

(iii) — " — <u>quasilinear</u> if it has the form

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

(iv) — " — <u>fully nonlinear</u> if there is a nonlinear dependence on the higest order derivatives.

<u>Boundary condition classification</u>:

- Dirichlet boundary condition $\quad u\big|_{\partial\Omega} = \varphi \quad$ for $\varphi \in C(\partial\Omega)$,
- Neumann — " — $\quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \psi \quad$ for $\psi \in C(\partial\Omega)$
- a linear combination of both Dirichlet and Neumann boundary conditions.

e.g. linear equations:

$$\begin{cases} \Delta u = \sum_{k=1}^{n} \dfrac{\partial^2 u}{\partial x_k^2} = 0 & \text{on } \Omega \quad (\text{Laplace/Harmonic}) \\ u_t = \Delta u & \text{on } \Omega \times \mathbb{R}_{>0} \quad (\text{Heat}) \\ u_{tt} = \Delta u & \text{on } \Omega \times \mathbb{R}_{>0} \quad (\text{wave}) \end{cases}$$

e.g. non-linear equations:

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = 0 & (\text{minimal surface; quasilinear}) \\ \det D^2 u = f & (\text{Monge-Ampère; fully nonlinear}) \end{cases}$$

e.g. the equation of the gravitational field in non-relativistic mechanics is given by

$$\Delta \varphi = 4\pi k \mu \quad \Rightarrow \quad \varphi = -k \int \frac{\mu \, dV}{R}$$

where μ = mass density of body

k = const.

(analogous to the Poisson equation for the electric potential)

In particular, for the potential of the field of a single particle of mass m:

$$\varphi = -\frac{km}{R}.$$

Consequently, the force F acting in the field on another particle of mass m' is equal to

$$F = -m' \frac{\partial \varphi}{\partial R} = -k \frac{m m'}{R^2}$$

(the law of attraction of Newton)