MOTIVATING EXAMPLES

§ Riemann's "foundations of complex analysis"

Riemann (1851) aimed to construct holomorphic functions $w = u + iv$, as a function of $z = x + iy$, that satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

by using a variational principle.

§ Dirichlet's Principle.

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and given $u_0 \in C(\partial\Omega)$ we want to find a function

$$u \in \mathcal{E}(\Omega; u_0) = \{ u \in C^2(\Omega) \cap C(\bar{\Omega}) : u = u_0 \text{ on } \partial\Omega \}$$

that minimises

$$E(w) = \frac{1}{2} \int_\Omega |Dw|^2 dx$$

for all $w \in \mathcal{E}(\Omega; u_0)$.

NB: if the minimiser u exists then it solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

(i.e. stationary points are harmonic)

§ Weierstraß counter-example

Let $J(u) = \int_{-1}^{1} x^2 u'(x)^2 dx$

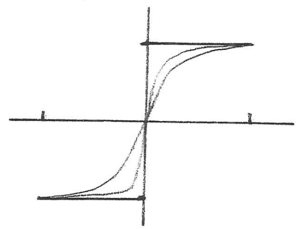
and take $\mathcal{E} = \{ u \in C^1[-1,1] : u(-1) = a, u(1) = b \}$, $a \neq b$.

Now construct

$$\boxed{u_\varepsilon(x) = \frac{a+b}{2} + \frac{b-a}{2} \frac{\arctan x/\varepsilon}{\arctan 1/\varepsilon}, \quad \varepsilon > 0.}$$

Then $u_\varepsilon \in \mathcal{E}$ for all $\varepsilon > 0$ and

$$0 \le J(u_\varepsilon) \le \int_{-1}^{1} (x^2 + \varepsilon^2) u_\varepsilon'(x)^2 dx,$$



As $u_\varepsilon'(x) = \dfrac{b-a}{2 \arctan 1/\varepsilon} \dfrac{\varepsilon}{x^2 + \varepsilon^2}$

we get

$$J(u_\varepsilon) \le \frac{\varepsilon}{2} \frac{(b-a)^2}{\arctan 1/\varepsilon}.$$

This implies

$$0 \le \inf_{u \in \mathcal{E}} J(u) \le \liminf_{\varepsilon \searrow 0} J(u_\varepsilon) = 0.$$

However if $J(u_{min}) = 0$ then $u_{min} = \text{const.}$ $\forall x \in [-1,1]$.
Thus $u_{min}(-1) = u_{min}(1) \Rightarrow u_{min} \notin \mathcal{E}$ (contradiction)

Remark. Weierstraß' critique of the Dirichlet principle put the existence proof into a Grundlagenkrise.

e.g. Consider the punctured unit disk $\Omega = B \setminus \{0\}$ and let

$$\mathcal{C} = \{ u \in C^2(\Omega) \cap C(\bar{\Omega}) : u(0) = 1, \ u|_{\partial B} = 0 \}$$

Now take a cut-off function $\eta \in C^\infty(\mathbb{R})$ s.t.

$$\eta(x) = \begin{cases} 1, & |x| \le 1 \\ 0 & |x| \ge 2 \end{cases}$$

and set

$$u_\varepsilon(x) = \begin{cases} 1 - \eta\left(\frac{\log |x|}{\log \varepsilon} \right), & 0 \le |x| \le 1 \\ 1, & x = 0 \end{cases}$$

(i.e. $u_\varepsilon = 1$ for $0 \le |x| \le \varepsilon^2$ and $u_\varepsilon = 0$ for $\varepsilon < |x| \le 1$)

Then $u_\varepsilon \in \mathcal{C}$ and (+)

$$0 \le \mathcal{E}(u_\varepsilon) = \frac{1}{2} \int_B |Du_\varepsilon|^2 dx \le \sup |D\eta|^2 \, \frac{2|B|}{(\log \varepsilon)^2} \int_{\varepsilon^2}^{\varepsilon} t^{n-3} dt .$$

Since $\mathcal{E}(u_\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$ we have $\inf_{u \in \mathcal{C}} \mathcal{E}(u) = 0$.

But $\nexists \, u_{min} \in \mathcal{C}$ s.t. $\mathcal{E}(u_{min}) = 0$!

(+) assuming $n \ge 3$.

§ Continuous nowhere differentiable functions

- Is $f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$ differentiable? (Riemann 1861, conjecture)

(NB: continuity follows by the convergence of the series)

- Weierstraß example

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

for $0 < a < 1$, $b > 1$ s.t. $ab \ge 1$ (cf. Hardy 1916)

Remark. If $a = b^{-\alpha}$, $0 < \alpha < 1$ then

(i) $\sum b^{-\alpha n} \cos(b^n \pi x) \in C^\alpha$ = Hölder cont.

and

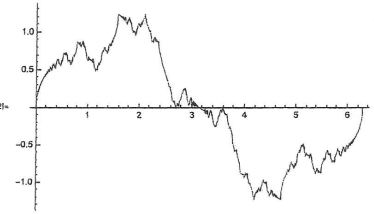
(ii) $\sum b^{-n} \cos(b^n \pi x) \in C^\alpha$ for all $\alpha < 1$ but is not Lipschitz continuous.

In[1]:= Evaluate@Sum[$\frac{1}{n^2}$ * Sin[n² x], {n, 1, 20}]

Out[1]= Sin[x] + $\frac{1}{4}$ Sin[4 x] + $\frac{1}{9}$ Sin[9 x] + $\frac{1}{16}$ Sin[16 x] + $\frac{1}{25}$ Sin[25 x] + $\frac{1}{36}$ Sin[36 x] + $\frac{1}{49}$ Sin[49 x] + $\frac{1}{64}$ Sin[64 x] + $\frac{1}{81}$ Sin[81 x] + $\frac{1}{100}$ Sin[100 x] + $\frac{1}{121}$ Sin[121 x] + $\frac{1}{144}$ Sin[144 x] + $\frac{1}{169}$ Sin[169 x] + $\frac{1}{196}$ Sin[196 x] + $\frac{1}{225}$ Sin[225 x] + $\frac{1}{256}$ Sin[256 x] + $\frac{1}{289}$ Sin[289 x] + $\frac{1}{324}$ Sin[324 x] + $\frac{1}{361}$ Sin[361 x] + $\frac{1}{400}$ Sin[400 x]

In[2]:= Plot[%, {x, 0, 2π}, PlotStyle → Thickness[Medium]]

Out[2]=



§ Fourier Series

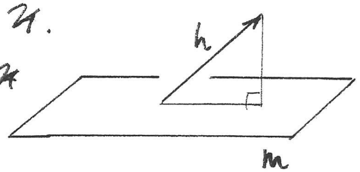
Theorem. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$. Denote by Ph the unique point in $\mathcal{M}$ s.t. $h - Ph \perp \mathcal{M}$. Then,

(a) P is a linear transform of $\mathcal{H}$.

(b) $\|Ph\| \le \|h\|$ for all $h \in \mathcal{H}$

(c) $P^2 = P$

(d) $\ker P = \mathcal{M}^\perp$, range$(P) = \mathcal{M}$.



Moreover we call P the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.

Def. A basis for $\mathcal{H}$ is a maximal orthonormal set

e.g. $\mathcal{H} = L^2[0, 2\pi]$ where $e_n \in \mathcal{H}$ def by $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$.

Then $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal set in $\mathcal{H}$.

Theorem. If $\mathcal{E}$ is an orthonormal set in $\mathcal{H}$, then the following statements are equivalent:

(i) $\mathcal{E}$ is a basis for $\mathcal{H}$.

(ii) If $h \in \mathcal{H}$ and $h \perp \mathcal{E}$ then $h = 0$.

(iii) $\mathrm{Conv}(\mathcal{E}) = \mathcal{H}$

(iv) any $h \in \mathcal{H}$ can be written as $h = \sum_{e \in \mathcal{E}} \langle h, e\rangle e$

(v) If $g, h \in \mathcal{H}$ then $\langle g, h\rangle = \sum_{e \in \mathcal{E}} \langle g, e\rangle \langle e, h\rangle$

(v) If $h \in \mathcal{H}$ then

$$\|h\|^2 = \sum_{e \in \mathcal{E}} |\langle h, e\rangle|^2 \quad \text{(Parseval's identity)}$$

e.g. If $e_n(t) = e^{int}$, $n \in \mathbb{Z}$, then $\{e_n : n \in \mathbb{Z}\}$ is a basis for $L^2_{\mathbb{C}}([0, 2\pi], \frac{dt}{2\pi})$ and

$$\hat{f}(n) = \langle f, e_n\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$$

so that the Fourier series of f is given by

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$$

and converges in the metric defined by the norm of $\mathcal{H}$.

Theorem. The map

$$\begin{cases} L^2_{\mathbb{C}}[0, 2\pi] \to \ell^2(\mathbb{Z}) \\ f \mapsto (\hat{f}(n)) \end{cases}$$

is a linear isometry.