FOURIER TRANSFORM   Lecture 4

Def. The <u>Fourier transform</u> (FT) on $\mathbb{R}^n$ is taken to be

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx \qquad \text{— (1)}$$

for "sufficiently nice" functions (so the integral converges).
Likewise, the <u>inverse Fourier transform</u> is given by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi. \qquad \text{— (2)}$$

§ Basic Properties.

If we denote $\boxed{D_j = -i\frac{\partial}{\partial x_j}}$ then integration by parts in (1) gives

$$\mathcal{F}(D_j f) = \xi_j\,\mathcal{F}(f).$$

By writing $x_j e^{-ix\cdot\xi} = i\frac{\partial}{\partial \xi_j} e^{-ix\cdot\xi}$ we get

$$\mathcal{F}(x^j f) = -D_j\,\mathcal{F}(f).$$

In general, for a multi-index $\beta = (\beta_1, \dots, \beta_n)$ we write

$$D^\beta = D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n}, \quad \xi^\beta = \xi_1^{\beta_1}\cdots\xi_n^{\beta_n},$$

so that we get

$$\begin{cases} \mathcal{F}(D^\beta f) = \xi^\beta\,\mathcal{F}(f) \\ \mathcal{F}(x^\beta f) = (-1)^{|\beta|} D^\beta \mathcal{F}(f) \end{cases}$$

(i.e. the FT converts derivatives into polynomials and vice versa)

e.g. the FT of the Gaussian is given by

$$\mathcal{F}(e^{-a|x|^2/2}) = \left(\frac{2\pi}{a}\right)^{n/2} e^{-|\xi|^2/2a}.$$

<u>Notation</u>. Also denote $\hat{f}(\xi) := \mathcal{F}(f)(\xi)$ "hat notation"

§ Test functions.

Q: On what class does the Fourier transform act in a well defined way?

Def. A function f on $\mathbb{R}^n$ is said to be of <u>rapid decrease</u> if for any multi-indices α, β we have

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < +\infty.$$

The set of such functions, denoted by $S(\mathbb{R}^n)$, is called the <u>Schwartz function class</u>.

<u>Remarks</u>. (i) The quantity $\|f\|_{\alpha,\beta}$ denotes a semi-norm.

(ii) We say a sequence (f_n) in $S(\mathbb{R}^n)$ converges to f if and only if

$$\|f_n - f\|_{\alpha,\beta} \to 0, \quad \forall\, \alpha, \beta.$$

<u>Exercise</u>. Show that if $\varphi \in S(\mathbb{R}^n)$ then $x^\alpha\varphi,\ D^\alpha\varphi \in S(\mathbb{R}^n)$.

Thm A. $\mathcal{F}$ is a continuous map from $S(\mathbb{R}^n)$ to itself.

Thm B. $\mathcal{F}^{-1}$ on $S(\mathbb{R}^n)$ is a two-sided inverse of $\mathcal{F}$.

Thm C. The FT preserves the L^2-norm of the Schwartz functions in $\mathbb{R}^n$, i.e.

$$\|\mathcal{F}f\|_{L^2} = (2\pi)^{n/2}\|f\|_{L^2}, \quad \forall f \in S(\mathbb{R}^n).$$

Proof of A. Given $\varphi \in S(\mathbb{R}^n)$ we want to show $\mathcal{F}\varphi \in S(\mathbb{R}^n)$

(i.e. $\mathcal{F}(\partial^\alpha x^\beta \varphi) = \xi^\alpha \partial_\xi^\beta \hat{\varphi}(\xi) \in C^\infty(\mathbb{R}^n)$, $\forall \alpha, \beta$)

By the exercise, $g = \partial^\alpha x^\beta \varphi \in S(\mathbb{R}^n)$. Hence $g \in L^1(\mathbb{R}^n)$.

Now as $|\hat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\,dx$ we have $\|\hat{f}\|_\infty = \sup|\hat{f}(\xi)| \le \|f\|_{L^1}$.

Thus $\mathcal{F}g \in L^\infty$ as required. (exercise: check continuity). ∎

Proof C.

The L^2 norm in Lhs hand is of the form

$$\int d\xi \left[\int e^{-iy\cdot\xi} f(y)\,dy\right]\left[\int e^{ix\cdot\xi}\overline{f(x)}\,dx\right]$$

$$= \lim_{\varepsilon\to 0}\int d\xi\, e^{-\varepsilon|\xi|^2/2}\left[\int e^{-iy\cdot\xi} f(y)\,dy\right]\left[\int e^{ix\cdot\xi}\overline{f(x)}\,dx\right]$$

$$= \lim_{\varepsilon\to 0}\left(\frac{2\pi}{\varepsilon}\right)^{n/2}\iint e^{-|x-y|^2/2\varepsilon} f(y)\overline{f(x)}\,dx\,dy, \quad \text{(by Fubini)}$$

$$= (2\pi)^n \int |f(x)|^2\,dx. \quad ∎$$

Proof of B. For any $\varphi \in S(\mathbb{R}^n)$,

$$\mathcal{F}^{-1}\circ\mathcal{F}(\varphi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} d\xi\, \underbrace{e^{ix\cdot\xi}\left(\int_{\mathbb{R}^n} dy\, e^{-iy\cdot\xi}\varphi(y)\right)}_{\text{there is no decay in } \xi\text{-direction}}$$

[Regularization]

$$= \lim_{\delta\to 0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} d\xi\, \boxed{e^{-\frac{\delta|\xi|^2}{2}}}\, e^{ix\cdot\xi}\left(\int_{\mathbb{R}^n} dy\, e^{-iy\cdot\xi}\varphi(y)\right)$$

(by the DCT, since $\hat{\varphi} = \mathcal{F}\varphi \in L^1$ and $e^{-\delta|\xi|^2/2} \le 1$)

$$= \lim_{\delta\to 0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} dy\, \varphi(y)\underbrace{\left(\int_{\mathbb{R}^n} d\xi\, e^{-\frac{\delta|\xi|^2}{2}} e^{-i(y-x)\cdot\xi}\right)}_{=\left(\frac{2\pi}{\delta}\right)^{n/2} e^{-\frac{|y-x|^2}{2\delta}}}$$

(by Fubini Theorem, since the integrand is L^1-function on $\mathbb{R}^{2n}$)

$$= \lim_{\delta\to 0}\frac{1}{(2\pi\delta)^{n/2}}\int_{\mathbb{R}^n} dy\, \varphi(y)\, e^{-\frac{|y-x|^2}{2\delta}}$$

$$= \lim_{\delta\to 0}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}}\varphi(x+\sqrt{\delta}z)\,dz, \quad z = \frac{y-x}{\sqrt{\delta}},$$

$$= \varphi(x)\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/2}\,dz$$

(since the integrand is uniformly bounded by $e^{-\frac{|z|^2}{2}}\|\varphi\|_\infty$, $\forall \delta > 0$, we can take the pointwise limit)

$$= \varphi(x) \quad ∎$$

§ The Fourier integral as the limit of the Fourier series.

The Fourier series for a function of period $2L$ can be written in the

$$f(t) = \sum_{n\in\mathbb{Z}} c_n e^{i\frac{n\pi}{L}t} \qquad \text{— } (*)$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{n\pi t}{L}}\, dt .$$

The sum $(*)$ converges in L^2, since $\{c_n\} \in \ell^2(\mathbb{Z})$.

Re-writing $(*)$ by inserting the formula for the coef. yields

$$f(t) = \sum_{n\in\mathbb{Z}} \left[\frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{n\pi}{L}t}\, dt\right] e^{i n\frac{\pi}{L}} .$$

Now define

$$\omega_n = \frac{n\pi}{L}, \quad \Delta_n\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L} \qquad \text{— } (\dagger)$$

so that the series can be re-written in the form

$$f(t) = \sum_{n\in\mathbb{Z}} \left[\frac{1}{2\pi}\int_{-L}^{L} f(t)\, e^{-i\omega_n t}\, dt\right] e^{i\omega_n t}\,\Delta_n\omega$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\,\hat{f}_n\, e^{i\omega_n t}\,\Delta_n\omega, \qquad \text{— } (**)$$

where $\hat{f}_n = \hat{f}_L(\omega_n) = \int_{-L}^{L} f(t)\, e^{-i\omega_n t}\, dt .$

Then let $L \to \infty$ and from $(\dagger)$ observe $\Delta_n\omega \to 0$ so that ω_n becomes a continuous variable.

Hence we find

$$\hat{f}(\omega) = \lim_{L\to\infty} \hat{f}_L(\omega_n) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt,$$

and $(**)$ becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega .$$

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