FOURIER TRANSFORM (PART 2) Lecture 5.

1. Convolutions.

The convolution of $f, g \in S(\mathbb{R}^n)$, denoted by $f*g$, is given by

$$f*g(x) = \int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy.$$

NB: $f*g = g*f$.

Remark. The convolution appears in PDE in the context of (1) translation invariance, (2) integral representations and (3) smooth approximations.

Theorem. For any $f, g \in S(\mathbb{R}^n)$, the Fourier transform of $f*g$ is the pointwise product of $\hat{f}$ and $\hat{g}$.

Proof. Write the FT of the convolution

$$\int dx\, e^{-ix\cdot\xi} \int f(x-y)\,g(y)\,dy$$

using $e^{-ix\cdot\xi} = e^{-i(x-y)\xi}\, e^{-iy\cdot\xi}$ and do change of variables. ∎

Corollary. The convolution is a continuous map from $S(\mathbb{R}^n)\times S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$.

Proof. Apply the inverse FT to get $f*g = \mathcal{F}^{-1}(\hat{f}\hat{g})$. ∎

2. Mollifiers

Let η be the function on $\mathbb{R}^n$ that satisfies

$$(1)\ \eta \in C_c^\infty(\mathbb{R}^n);\quad (2)\ \eta \geq 0;\quad (3)\ \int_{\mathbb{R}^n} \eta(x)\,dx = 1,$$

and consider the scaling defined by

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\,\eta\left(\frac{x}{\varepsilon}\right).$$

Theorem. (Approximate identity)

(i) if $f \in C^0$ then $f*\eta_\varepsilon \to f$ uniformly

(ii) if f is continuous at x, then $f*\eta_\varepsilon(x) \to f(x)$

(iii) if $f \in L^p$, $1 \leq p < \infty$, then $f*\eta_\varepsilon \to f$ in L^p.

Proof. (See Evan's PDE appendix)

Remark (i) Note that $\lim_{\varepsilon\to 0} \eta_\varepsilon(x) = \delta(x)$ [Dirac delta "function".

(ii) There is no identity element under convolution for functions.

3. Distributions

Def. The space of tempered distributions, denoted by $S'(\mathbb{R}^n)$, is the set of continuous linear functionals on $S(\mathbb{R}^n)$.

(i.e. $u \in S'(\mathbb{R}^n)$ if $u : S(\mathbb{R}^n) \to \mathbb{C}$ is linear map)

We will write $u(\varphi)$, or $\langle u, \varphi \rangle$, for the pairing of $u \in S'(\mathbb{R}^n)$ with $\varphi \in S(\mathbb{R}^n)$.

NB: The space has the property that $u(\phi_n) \to u(\phi)$ whenever $\phi_n \to \phi$ in $S(\mathbb{R}^n)$.

e.g. (1) if $f \in L^1$ then $\boxed{T_f : \varphi \mapsto \int f \phi \, dx}$ is a distribution.

(2) for any multi-index β and $x_0 \in \mathbb{R}^n$,

$$\boxed{\varphi \mapsto D^\beta \varphi(x_0)}$$

is a distribution.

(3) The Dirac delta "function" (as a distribution) is defined by

$$\boxed{\delta : \varphi \mapsto \varphi(0).}$$

Remark (1) We want to enlarge the notion of pointwise functions to generalise the notion of a derivative.

(2) We use a dual argument to define the Fourier transform and to differentiate a distribution.

e.g. if $f \in C^1$ is of polynomial growth, then

$$\int_{\mathbb{R}^n} \partial_{x_j} f(x) \varphi(x) \, dx = - \int_{\mathbb{R}^n} f(x) \partial_{x_j} \varphi(x) \, dx.$$

Def. For any $u \in S'(\mathbb{R}^n)$ we define $\partial_{x_j} u \in S'(\mathbb{R}^n)$ by letting

$$\langle \partial_{x_j} u, \varphi \rangle := \langle u, -\partial_{x_j} \varphi \rangle$$

for all $\varphi \in S(\mathbb{R}^n)$.

In general, for any multi-index α, we have

$$\boxed{\langle \partial^\alpha u, \varphi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle}$$

for all $\varphi \in S(\mathbb{R}^n)$.

e.g. $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$.

e.g. for the Heaviside function $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ we have

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(x) \, dx = \varphi(0)$$

Thus $H' = \delta$.

Def. (multiplication by smooth functions) (†)

If $f \in C^\infty(\mathbb{R}^n)$ and $u \in S'(\mathbb{R}^n)$ then the product fu is defined to be $\langle fu, \varphi\rangle := \langle u, f\varphi\rangle$, $\forall \varphi \in S(\mathbb{R}^n)$.

e.g. $\langle x^\alpha u, \varphi\rangle = \langle u, x^\alpha \varphi\rangle$

e.g. $\langle f\delta, \varphi\rangle = f(0)\langle \delta, \varphi\rangle = \langle f(0)\delta, \varphi\rangle$

Def. (Fourier transform of distributions)

For every $u \in S'(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}u$ is given by $\langle \mathcal{F}u, \varphi\rangle := \langle u, \mathcal{F}\varphi\rangle$, $\forall \varphi \in S(\mathbb{R}^n)$.

e.g. $\langle \mathcal{F}\delta, \varphi\rangle = \langle \delta, \hat{\varphi}\rangle = \hat{\varphi}(0) = \int \varphi(x)\,dx = \langle 1, \varphi\rangle$.

Def. (Convolution of distributions

The convolution of a distribution $u \in S'(\mathbb{R}^n)$ with $\varphi \in S(\mathbb{R}^n)$ is given by $u * \varphi(x) = \langle u, R\tau_x\varphi\rangle$, $\forall \varphi \in S(\mathbb{R}^n)$, where $\tau_x\varphi(y) = \varphi(y-x)$ and $R\varphi(x) = \varphi(-x)$.

Remark. $u * \varphi \in C^\infty$ but may not be Schwartz.
If $u \in S'(\mathbb{R}^n)$ has compact support, then $u * \varphi \in S'(\mathbb{R}^n)$.

(†) NB: The multiplication of two arbitrary distributions is impossible! (e.g. cannot do/make sense of $\delta \cdot \delta$) cf. Schwartz 1954.

Def. (Convergence of distributions).

We say a sequence (u_j) in $S'(\mathbb{R}^n)$ converges to $u \in S'(\mathbb{R}^n)$ if $\langle u_j, \varphi\rangle \to \langle u, \varphi\rangle$ for all $\varphi \in S(\mathbb{R}^n)$.
(i.e. in the "weak sense").

Theorem (convolution properties)

If $u \in S'(\mathbb{R}^n)$ and $\varphi, \psi \in S(\mathbb{R}^n)$ then

(i) $u * \varphi \in C^\infty(\mathbb{R}^n)$

(ii) $\mathrm{supp}(u*\varphi) \subseteq \mathrm{supp}(u) + \mathrm{supp}(\varphi)$

(iii) $\partial^\alpha(u*\varphi) = \partial^\alpha u * \varphi = u * \partial^\alpha\varphi$.

(iv) $(u*\varphi)*\psi = u*(\varphi*\psi)$

Theorem (Regularisation of distributions)

Let $u \in S'(\mathbb{R}^n)$ and η_ε be the approximate identity. Then $u * \eta_\varepsilon \to u$ in $S'(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Proof. Note that $\langle u, \varphi\rangle = (u * R\varphi)(0)$.

Hence $\langle u*\eta_\varepsilon, \varphi\rangle = (u*\eta_\varepsilon)*R\varphi(0) = u*(\eta_\varepsilon * R\varphi)(0)$

But since η_ε is the approximate identity,

$$\eta_\varepsilon * R\varphi \to R\varphi \text{ in } S(\mathbb{R}^n).$$

Thus $\lim_{\varepsilon\to 0}\langle u*\eta_\varepsilon, \varphi\rangle = \lim_{\varepsilon\to 0} u*(\eta_\varepsilon * R\varphi)(0) = u * R\varphi(0)$
$= \langle u, \varphi\rangle$ ∎

L. C. Evans "Partial Differential Equations" (Appendix)

C.5. Convolution and smoothing.

We next introduce tools that will allow us to build smooth approximations to given functions.

NOTATION. If $U \subset \mathbb{R}^n$ is open and $\epsilon > 0$, we write

$$U_\epsilon := \{x \in U \mid \operatorname{dist}(x, \partial U) > \epsilon\}.$$

DEFINITIONS. (i) *Define* $\eta \in C^\infty(\mathbb{R}^n)$ *by*

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

the constant $C > 0$ *selected so that* $\int_{\mathbb{R}^n} \eta \, dx = 1$.

(ii) *For each* $\epsilon > 0$, *set*

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$

We call η the *standard mollifier*. The functions η_ϵ are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon \, dx = 1, \ \operatorname{spt}(\eta_\epsilon) \subset B(0, \epsilon).$$

DEFINITION. *If* $f : U \to \mathbb{R}$ *is locally integrable, define its* mollification

$$f^\epsilon := \eta_\epsilon * f \quad \text{in } U_\epsilon.$$

That is,

$$f^\epsilon(x) = \int_U \eta_\epsilon(x-y) f(y) \, dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y) \, dy$$

for $x \in U_\epsilon$.

THEOREM 7 (Properties of mollifiers).

(i) $f^\epsilon \in C^\infty(U_\epsilon)$.

(ii) $f^\epsilon \to f$ *a.e. as* $\epsilon \to 0$.

(iii) *If* $f \in C(U)$, *then* $f^\epsilon \to f$ *uniformly on compact subsets of* U.

(iv) *If* $1 \leq p < \infty$ *and* $f \in L^p_{\mathrm{loc}}(U)$, *then* $f^\epsilon \to f$ *in* $L^p_{\mathrm{loc}}(U)$.

Proof. 1. Fix $x \in U_\epsilon$, $i \in \{1, \dots, n\}$, and h so small that $x + he_i \in U_\epsilon$. Then

$$\begin{aligned} \frac{f^\epsilon(x+he_i) - f^\epsilon(x)}{h} &= \frac{1}{\epsilon^n} \int_U \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y) \, dy \\ &= \frac{1}{\epsilon^n} \int_V \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y) \, dy \end{aligned}$$

for some open set $V \subset\subset U$. As

$$\frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] \to \frac{1}{\epsilon}\eta_{x_i}\left(\frac{x-y}{\epsilon}\right)$$

uniformly on V, the partial derivative $f^\epsilon_{x_i}(x)$ exists and equals

$$\int_U \eta_{\epsilon, x_i}(x-y) f(y) \, dy.$$

A similar argument shows that $D^\alpha f^\epsilon(x)$ exists, and

$$D^\alpha f^\epsilon(x) = \int_U D^\alpha \eta_\epsilon(x-y) f(y) \, dy \quad (x \in U_\epsilon),$$

for each multiindex α. This proves (i).

2. According to Lebesgue's Differentiation Theorem (§E.4),

$$\lim_{r \to 0} ⨍_{B(x,r)} |f(y) - f(x)| \, dy = 0 \tag{4}$$

for a.e. $x \in U$. Fix such a point x. Then

$$\begin{aligned}
|f^\epsilon(x) - f(x)| &= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y)[f(y) - f(x)]\,dy \right| \\
&\leq \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{x-y}{\epsilon}\right) |f(y) - f(x)|\,dy \\
&\leq C \fint_{B(x,\epsilon)} |f(y) - f(x)|\,dy \to 0 \quad \text{as } \epsilon \to 0,
\end{aligned}$$

by (4). Assertion (ii) follows.

3. Assume now $f \in C(U)$. Given $V \subset\subset U$, we choose $V \subset\subset W \subset\subset U$ and note that f is uniformly continuous on W. Thus the limit (4) holds uniformly for $x \in V$. Consequently the calculation above implies $f^\epsilon \to f$ uniformly on V.

4. Next, assume $1 \leq p < \infty$ and $f \in L^p_{\text{loc}}(U)$. Choose an open set $V \subset\subset U$ and, as above, an open set W so that $V \subset\subset W \subset\subset U$. We claim that for sufficiently small $\epsilon > 0$

$$\|f^\epsilon\|_{L^p(V)} \leq \|f\|_{L^p(W)}. \tag{5}$$

To see this, we note that if $1 \leq p < \infty$ and $x \in V$,

$$\begin{aligned}
|f^\epsilon(x)| &= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y) f(y)\,dy \right| \\
&\leq \int_{B(x,\epsilon)} \eta_\epsilon^{1-1/p}(x-y)\eta_\epsilon^{1/p}(x-y)|f(y)|\,dy \\
&\leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,dy \right)^{1-1/p} \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dy \right)^{1/p}.
\end{aligned}$$

Since $\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,dy = 1$, this inequality implies

$$\begin{aligned}
\int_V |f^\epsilon(x)|^p\,dx &\leq \int_V \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p\,dy \right) dx \\
&\leq \int_W |f(y)|^p \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)\,dx \right) dy = \int_W |f(y)|^p\,dy,
\end{aligned}$$

provided $\epsilon > 0$ is sufficiently small. This is (5).

5. Now fix $V \subset\subset W \subset\subset U$, $\delta > 0$, and choose $g \in C(W)$ so that

$$\|f - g\|_{L^p(W)} < \delta.$$

Then

$$\begin{aligned}
\|f^\epsilon - f\|_{L^p(V)} &\leq \|f^\epsilon - g^\epsilon\|_{L^p(V)} + \|g^\epsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)} \\
&\leq 2\|f - g\|_{L^p(W)} + \|g^\epsilon - g\|_{L^p(V)} \quad \text{by (5)} \\
&\leq 2\delta + \|g^\epsilon - g\|_{L^p(V)}.
\end{aligned}$$

Since $g^\epsilon \to g$ uniformly on V, we have $\limsup_{\epsilon\to 0} \|f^\epsilon - f\|_{L^p(V)} \leq 2\delta$. □