LAPLACE EQUATION

Let $u \in C^2(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ and recall that the Laplacian of u is given by

$$\Delta u = \operatorname{div}(Du).$$

We say u is **harmonic** in Ω if $\Delta u = 0$.

From the divergence theorem with $w = Du$ we have

$$\int_\Omega \operatorname{div} Du \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\sigma. \qquad \text{—}(*)$$

Theorem (mean-value property) [MVT]

Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ $(\geq, \leq)$ in Ω. Then for all $B_R = B_R(x_0) \subset\subset \Omega$ we have

$$u(x) = (\leq, \geq) \fint_{\partial B(x,R)} u(y)\,d\sigma(y) = \fint_{B(x,R)} u(y)\,dy.$$

(Here: $\fint_E f = \frac{1}{|E|}\int_E f$ denotes the average integral

and $V \subset\subset U$ denotes "V compactly contained in U", i.e. $V \subset \bar{V} \subset U$ with $\bar{V}$ compact)

Proof. For $0 < r < R$ let

$$\phi(r) = \fint_{\partial B(x,r)} u(y)\,d\sigma(y) = \fint_{\partial B(0,1)} u(x+rz)\,d\sigma(z).$$

Then

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz)\cdot z \, d\sigma(z)$$

$$= \fint_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} \, d\sigma(y)$$

$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, d\sigma(y), \qquad \text{NB: } \nu = \frac{y-x}{|y-x|}$$

$$= (\geq, \leq)\ 0 \quad \text{by } (*).$$

Consequently

$$\phi(r) = (\leq, \geq)\ \phi(R)$$

and since

$$\lim_{r\to 0} \fint_{\partial B(x,r)} u(y)\,d\sigma(y) = u(x)$$

the first relation follows.

To get the second identity, we write

$$\int_{B(x,R)} u(y)\,dy = \int_0^R \left(\int_{\partial B(x,s)} u \, d\sigma \right) ds \qquad \text{(coarea formula)}$$

$$= (\leq, \geq)\ |B_R(x)|\, u(x). \qquad \blacksquare$$

Theorem (Converse to mean-value property)

If $u \in C^2(\Omega)$ satisfies $u(x) = \fint_{\partial B(x,r)} u \, d\sigma$ for each $B(x,r) \subset\subset \Omega$, then u is harmonic.

Proof. If $\Delta u \not\equiv 0$ then there exists some $B(x,r) \subset \Omega$ s.t. $\Delta u > 0$ within $B(x,r)$.

However, for ϕ as above,

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy > 0,$$

a contradiction. ∎

Theorem (Strong maximum principle)

Let $\Delta u \geq 0 \; (\leq 0)$ in Ω and suppose there exists a point $y \in \Omega$ for which $u(y) = \sup_\Omega u \; (= \inf_\Omega u)$. Then u is constant.

Consequently, harmonic functions cannot assume an interior maximum value unless it's constant.

Proof. Let Δu in Ω, let $M = \sup_\Omega u$, and define $\Omega_M = \{x \in \Omega : u(x) = M\}$.

By assumption Ω_M is not empty. Furthermore since u is continuous, Ω_M is closed relative to Ω.

Now let $z \in \Omega_M$ and apply the mean value property in a ball $B(z,r) \subset\subset \Omega$ to get

$$M = u(z) = \fint_{B(z,r)} u \leq M,$$

so that $u = M$ in $B(z,r)$. Consequently Ω_M is both open and relatively closed in Ω, and thus is equal to Ω if Ω is connected. ∎

Theorem (Weak maximum principle)

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $\Delta u \geq 0 \; (\leq 0)$ in Ω. Then, provided Ω is bounded,

$$\sup_\Omega u = \sup_{\partial\Omega} u \quad \left(\inf_\Omega u = \inf_{\partial\Omega} u\right).$$

Consequently, for harmonic u,

$$\inf_{\partial\Omega} u \leq u(x) \leq \sup_{\partial\Omega} u, \quad x \in \Omega.$$

Theorem (uniqueness of classical Dirichlet problem)

Let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\begin{cases} \Delta u = \Delta v & \text{in } \Omega \\ u = v & \text{on } \partial\Omega \end{cases}$$

Then $u = v$ in Ω.

Proof. Let $w = u - v$.

Then $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$.

By the previous theorem $w = 0$ in Ω. ∎

§ Interior Estimates of Derivatives.

If u is harmonic on Ω and $B_R(y) \subset\subset \Omega$, then the gradient Du is also harmonic in Ω and it follows by the MVT that

$$Du(y) = \fint_{B_R} Du \underset{\substack{\uparrow \\ \text{divergence theorem}}}{=} \frac{1}{|B_R|} \int_{\partial B_R} u\, \nu \, d\sigma .$$

Hence we get

$$|Du(y)| \le \frac{n}{R} \sup_{\partial B_R(y)} |u|$$

and hence

$$|Du(y)| \le \frac{n}{d_y} \sup_{\Omega} |u|, \qquad \text{—— } (*)$$

where $d_y = \operatorname{dist}(y, \partial\Omega)$.

Theorem. Let u be harmonic in Ω, then for any $\Omega' \subset\subset \Omega$ and any multi-index α we have

$$\sup_{\Omega'} |D^\alpha u| \le \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_{\Omega} |u|,$$

where $d = \operatorname{dist}(\Omega', \partial\Omega)$.

Proof. By induction we :

Claim. For any $k = 0,1,2,\dots$ there exists $c_{n,k} > 0$ such that

$$|D^\alpha u(y)| \le \frac{c_{n,k}}{R^k} \sup_{B_R(y)} |u|$$

for each $B_R(y) \subset\subset \Omega$ and any multi-index α of order $|\alpha| = k$.

Proof of claim. For $k = 0, 1$ we use the above MVT result (*).

Now assume $k \ge 2$. Fix $B_R(y) \subset\subset \Omega$ and let $|\alpha| = k$.

Then $D^\alpha u = (D^\beta u)_{x_j}$ for some $j = 1, \dots, n$ and $|\beta| = k-1$.

By the MVT on $B(y, \frac{R}{k})$ we get

$$|D^\alpha u(y)| \le \frac{nk}{R} \sup_{\partial B(y, \frac{R}{k})} |D^\beta u| .$$

If $x \in \partial B(y, R/k)$ then $B(x, (\frac{k-1}{k})R) \subset B(y, R)$ and apply the claim on $B(x, (\frac{k-1}{k})R)$ for $k-1$ [induction hypothesis] to get

$$|D^\beta u(x)| \le \frac{c_{n,k-1}}{R^k} \sup_{B_R(y)} |u| .$$

Therefore

$$|D^\alpha u(y)| \le \frac{c_{n,k}}{R^{k+1}} \sup_{B_R(y)} |u| .$$

∎

§ Harnack Inequality

Theorem. Let $u \ge 0$ be harmonic in Ω. Then for any bounded subdomain $\Omega' \subset\subset \Omega$, there exists $C = C(n, \Omega', \Omega) > 0$ s.t.

$$\sup_{\Omega'} u \le C \inf_{\Omega'} u .$$

Proof. Let $y \in \Omega$ and $B_{4R}(y) \subset \Omega$. Then for points $x_1, x_2 \in B_R(y)$, the MVT (NB: $\Delta u = 0$) yields

$$u(x_1) = \fint_{B_R(x_1)} u \, dx \le \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} u \, dy$$

$$u(x_2) = \fint_{B_{3R}(x_2)} u \, dx \ge \frac{1}{\omega_n (3R)^n} \int_{B_{2R}(y)} u \, dx .$$

Consequently,

$$\sup_{B_R(y)} u \le 3^n \inf_{B_R(y)} u . \qquad (*)$$

Now let $\Omega' \subset\subset \Omega$ and choose $x_1, x_2 \in \bar{\Omega}'$ s.t. $u(x_1) = \sup_{\Omega'} u$, $u(x_2) = \inf_{\Omega'} u$.

Let $\Gamma \subset \bar{\Omega}'$ be a line joining x_1 & x_2 and choose R s.t. $4R < \operatorname{dist}(\Gamma, \partial\Omega)$.

By the Heine-Borel thm, Γ can be covered by a finite number N of balls of radius R.

Applying (*) on each ball then implies $u(x_1) \le 3^{nN} u(x_2)$. ∎