GREENS REPRESENTATION

(NB: presume existence)

Laplace's equation has the radial symmetric solution

$$\begin{cases} r^{2-n}, & n \geq 3 \\ \log r, & n = 2 \end{cases}$$

where r is the radial distance from a fixed point.

We thus fix a point $y \in \Omega$ and introduce the normalised fundamental solution to the Laplace equation:

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x-y|^{2-n}, & n \geq 3 \\ \frac{1}{2\pi} \log |x-y|, & n = 2 \end{cases}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})}$ = volume of the n-dim unit ball.

By a direct computation we have

$$D_i \Gamma(x-y) = \frac{1}{n\omega_n} (x_i - y_i)|x-y|^{-n} \ ;$$

$$D_{ij} \Gamma(x-y) = \frac{1}{n\omega_n} \left(|x-y|^2 \delta_{ij} - n(x_i - y_i)(x_j - y_j) \right) |x-y|^{-n-2} .$$

The singularity at $x = y$ prevents us from using Γ in Green's second identity.

To overcome this we replace Ω by $\Omega \setminus \overline{B_\rho}$, where $B_\rho = B_\rho(y)$ for sufficiently small $\rho > 0$.

We can then conclude

$$\int_{\Omega \setminus B_\rho} \Gamma \Delta u \, dx = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) d\sigma + \int_{\partial B_\rho} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) d\sigma \quad -(\dagger)$$

Now

$$\int_{\partial B_\rho} \Gamma \frac{\partial u}{\partial \nu} d\sigma = \Gamma(\rho) \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} d\sigma$$

$$\leq n\omega_n \rho^{n-1} \Gamma(\rho) \sup_{B_\rho} |Du| \to 0 \quad \text{as } \rho \to 0,$$

and

$$\int_{\partial B_\rho} u \frac{\partial \Gamma}{\partial \nu} d\sigma = -\Gamma'(\rho) \int_{\partial B_\rho} u \, d\sigma \quad \text{(NB: } \nu \text{ is outer normal to } \Omega \setminus B_\rho)$$

$$= \frac{-1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho} u \to -u(y) \quad \text{as } \rho \to 0.$$

Thus by taking $\varepsilon \to 0$ in (†) we arrive at

Green's representation formula:

$$\boxed{\begin{aligned} u(y) &= \int_{\partial\Omega} u \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu}\, d\sigma \\ &\quad + \int_{\Omega} \Gamma(x-y)\,\Delta u\, dx, \quad y \in \Omega. \end{aligned}} \qquad (\ddagger)$$

NB: for integrable f, the expression

$$\int_{\Omega} \Gamma(x-y) f(x)\, dx$$

is called the Newton potential with density f.

Remark. From (‡) we can solve for u provided we know that

(i) Δu in Ω

(ii) u, $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$.

However the normal derivative $\frac{\partial u}{\partial \nu}$ is unknown to us.

Remark. Suppose $\Gamma(x,y) = \Gamma(x-y)$ solves $\Delta\Gamma(x,y) = -\delta(x-y)$ on $\mathbb{R}^n$. Then $u(x) = \int \Gamma(x,y) f(y)\, dy$, for given f, solves

$$\Delta_x u(x) = \int (\Delta_x \Gamma(x,y)) f(y)\, dy = \int \delta(x-y) f(y)\, dy = f(x)$$

on $\mathbb{R}^n$.

To account for this, consider $h \in C^1(\bar\Omega) \cap C^2(\Omega)$ satisfying

$$\Delta h = 0 \quad \text{in } \Omega.$$

Then by Green's second identity we obtain

$$-\int_{\partial\Omega} \left(u \frac{\partial h}{\partial \nu} - h \frac{\partial u}{\partial \nu} \right) d\sigma = \int_{\Omega} h\,\Delta u\, dx \qquad (\ddagger)$$

Now write $G = \Gamma + h$ and add (‡) with (‡) to obtain the more general version of the Green's representation formula:

$$u(y) = \int_{\partial\Omega} \left(u \frac{\partial G}{\partial \nu} - G \frac{\partial u}{\partial \nu} \right) d\sigma + \int_{\Omega} G\,\Delta u\, dx.$$

Now if in addition $G = 0$ (†) on $\partial\Omega$ we obtain

$$\boxed{u(y) = \int_{\partial\Omega} u \frac{\partial G}{\partial \nu}\, d\sigma + \int_{\Omega} G\,\Delta u\, dx}$$

and the function $G(x,y) = G$ is called the (Dirichlet) Green's function for the domain Ω.

It gives a representation for $C^1(\bar\Omega) \cap C^2(\Omega)$ - harmonic functions in terms of its boundary values.

(†) i.e. $\Gamma + h = 0$ on $\partial\Omega$ so that h solves

$$\begin{cases} \Delta h = 0 & \text{in } \Omega \\ h = -\Gamma & \text{on } \partial\Omega. \end{cases}$$

§ Physical meaning of Green's function

$$\begin{cases} \Delta_x G(x,y) = -\delta(x-y), & x \in \Omega \\ G = 0, & \text{on } \partial\Omega \end{cases}$$

$G(x,y)$ = Coulomb potential generated inside the grounded conducting surface $\partial\Omega$ by the charge of $+\frac{1}{4\pi}$ at the point $y \in \Omega$.