GREENS FUNCTION (PART 2)

§ Constructing Green's function by "method of images".

In general the Green's function depends both on the linear operator and the boundary conditions.

In deriving the Green's function we needed to explicitly solve

$$\begin{cases} \Delta h = 0 & \text{in } \Omega \\ h = -\Gamma & \text{on } \partial\Omega \end{cases}.$$

For domains with simple geometry, we can use a geometric reflection principle / technique.

§ Green's function for a ball.

We do a reflection principle through the sphere.

Definition. For $x \in \mathbb{R}^n \setminus 0$ the point $\bar{x} = \frac{x}{|x|^2}$ is called the point dual to x w.r.t. $\partial B(0,1)$.

The mapping $x \mapsto \bar{x}$ is an inversion through the unit sphere $\partial B(0,1)$.

For fixed $x \in B(0,1)$ we want to find the "correction" function $h = h^x(y)$ that solves

$$\begin{cases} \Delta h = 0 & \text{in } B(0,1) \\ h = -\Gamma(y-x) & \text{on } \partial B(0,1) \end{cases}.$$

Then the Green's function will be

$$G(x,y) = \Gamma(y-x) + h^x(y).$$

The idea now is to "invert the singularity" from $x \in B(0,1)$ to $\bar{x} \notin B(0,1)$.

NB: For $n \geq 3$ we have $\begin{cases} \text{(i) } y \mapsto \Gamma(y-\bar{x}) \text{ harmonic } y \neq \bar{x} \\ \text{(ii) } y \mapsto |x|^{2-n}\Gamma(y-\bar{x}) \text{ harmonic also} \end{cases}$

Hence $h = h^x(y) = -\Gamma(|x|(y-\bar{x}))$ is harmonic in $B(0,1)$.

Moreover, if $y \in \partial B(0,1)$ and $x \neq 0$ we note that

$$|x|^2|y-\bar{x}|^2 = |x|^2\left(|y|^2 - \frac{2x\cdot y}{|x|^2} + \frac{1}{|x|^2}\right)$$

$$= |x|^2 - 2y\cdot x + 1 = |x-y|^2.$$

Consequently $h^x(y) = -\Gamma(y-x)$ for $y \in \partial B(0,1)$.

In general, for a ball $B_R = B_R(0)$ with $x \in B_R$, $x \neq 0$, we let

$$\bar{x} = \frac{R^2}{|x|^2}\, x$$

denote the inverse point w.r.t. ∂B_R

(if $x = 0$, take $\bar{x} = \infty$).

It follows that the corresponding Green's function for B_R is given by

$$G(x,y) = \begin{cases} \Gamma(|x-y|) - \Gamma\left(\frac{|y|}{R}\,|x-\bar{y}|\right), & y \neq 0 \\ \Gamma(|x|) - \Gamma(R), & y = 0 \end{cases}$$

NB: $G(x,y) = G(y,x)$ for any $x, y \in \bar{B}_R$.

Furthermore, a direct calculation shows that at $x \in \partial B_R$ the normal derivative is given by

$$\frac{\partial G}{\partial \nu} = \frac{R^2 - |y|^2}{n\omega_n R}\, |x-y|^{-n} \geq 0.$$

Hence if $u \in C^2(B_R) \cap C^1(\bar{B}_R)$ is harmonic, we get

$$\boxed{u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{u \, d\sigma_x}{|y-x|^n}} \longleftarrow (\dagger)$$

(the so-called "Poisson integral formula").

Remark. An approximation argument shows that the Poisson integral formula continues to hold for

$$u \in C^2(B_R) \cap C^0(\bar{B}_R).$$

Also if $y = 0$ then $u(0) = \fint_{\partial B_R} u \, d\sigma$ (MVT).

<u>Theorem (Poisson's formula for ball)</u> "Dirichlet problem for balls."

Assume $\varphi \in C^0(\partial B_R)$ and u is given by $(\dagger)$.

Then

(i) $u \in C^\infty(B_R)$;

(ii) $\Delta u = 0$ in B_R;

(iii) $\displaystyle\lim_{\substack{x \to x_0 \\ x \in B_R}} u(x) = \varphi(x_0)$, $\forall x_0 \in \partial B_R$.