consider $\Delta u = f$ in $B_1(0) \subset \mathbb{R}^n$

with $\omega_f(t) = \sup\{ |f(x) - f(y)| : |x-y| < t \}$.

<u>Claim</u>. For $x, y \in B_{1/2}(0)$,

$$|D^2u(x) - D^2u(y)| \le c\left[d \sup_{B_1} |u| + \int_0^d \frac{\omega_f(t)}{t}\,dt + d\int_d^1 \frac{\omega_f(t)}{t^2}\,dt \right]$$

where $c = c(n) > 0$ and $d = |x-y|$.

<u>Proof</u>. Denote by $B_k = B(0, \rho^k)$ where $\rho = 1/2$.

For $k \in \mathbb{N}_0$, let u_k be the solution to

$$\begin{cases} \Delta u_k = f(0) & \text{in } B_k \\ u_k = u & \text{on } \partial B_k \end{cases}$$

NB: $\Delta(u_k - u) = f(0) - f$.

By applying the maximum principle to

$$v_k^{\pm} = u_k - u \pm \frac{\omega(\rho^k)}{2n}|x|^2$$

we find that

<u>claim 1</u>: $|u_k - u| \le C\,\omega(\rho^k)\rho^{2k}$.

NB: $\Delta v_k^+ = f(0) - f(x) + \omega(\rho^k) \ge 0$

$$\Rightarrow \sup_{B_k} v_k^+ = \sup_{\partial B_k} v_k^+ = C\,\omega(\rho^k)\rho^{2k}$$

$$\Rightarrow u_k - u \le v_k^+ \le C\,\omega(\rho^k)\rho^{2k}. \quad \text{(likewise for } v_k^-\text{)}$$

It follows that

$$\|u_k - u\|_{L^\infty(B_k)} \le C\rho^{2k}\omega(\rho^k) \qquad \text{— (1)}$$

Hence,

$$\begin{aligned} \|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} &\le \|u_k - u\|_{L^\infty(B_{k+1})} + C\rho^{2k+1}\omega(\rho^{k+1}) \\ &\le \|u_k - u\|_{L^\infty(B_k)} + \text{———} \\ &\le C\rho^{2k}\omega(\rho^k), \end{aligned} \qquad \text{— (2a)}$$

since $\omega(\cdot)$ is non-decreasing and $\rho^{k+2} < \rho^{k+1} < \rho^k$.

As $u_k - u_{k+1}$ is harmonic in B_{k+1}, we get by interior estimates that for $m \in \mathbb{N}_0$:

$$\|D^m(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \le C_{n,m}\,\rho^{-km}\,\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \qquad \text{— (2b)}$$

(since $\mathrm{dist}(B_{k+2}, \partial B_{k+1}) = \rho(1-\rho)\rho^k$

In particular, we get

$$\left.\begin{aligned} \|D(u_h - u_{h+1})\|_{L^\infty(B_{h+2})} &\le C\rho^h\,\omega(\rho^h) \\ \|D^2(u_h - u_{h+1})\|_{L^\infty(B_{h+2})} &\le C\,\omega(\rho^h) \end{aligned}\right\} \quad (3)$$

Claim 2. If q is the quadratic part of u, then

(i) $u_h - q$ is harmonic in B_h

(ii) $u_h - q = o(\rho^{2h})$ in B_h

(iii) $Du_h(0) \to Du(0)$ and

$D^2u_h(0) \to D^2u(0)$ as $h \to \infty$.

Proof of Claim 2. Let q be the Taylor polynomial of degree 2 at $x=0$, i.e.

$$\begin{aligned} u(x) &= u(0) + Du(0)(x) + \tfrac{1}{2} D^2u(0)(x,x) + o(|x|^2) \\ &= q(x) + o(|x|^2) \end{aligned}$$

(using $u \in C^2$ assumption).

and let

$$\begin{cases} \bar{u}_h = u_h - q \\ \bar{u} = u - q \end{cases} \qquad \left[\begin{aligned} \text{NB: } \Delta\bar{u}_h &= \Delta u_h - \Delta q \\ &= f(0) - \Delta u(0) = 0 \\ \Rightarrow\ &\bar{u}_h \text{ is harmonic on } B_h. \end{aligned}\right]$$

Then by the MVT,

$$|D^2\bar{u}_h(0)| \underset{(i)}{\le} C_n \rho^{-h} \sup_{\partial B_h} |D\bar{u}| \underset{(ii)}{<} \epsilon\rho^h,$$

since,

(i) $\bar{u}_h = u = o(\rho^{2h})$ on ∂B_h

(ii) $D\bar{u} = D[O(|x|^3)] = O(|x|^2)$

Therefore $D^2\bar{u}_h(0) \to 0$ as $h \to \infty$. ∎

- We now observe that the RHS of the original claim depends only on the difference of x and y, hence we can take $x = z \in B_{1/2}$ and $y = 0$ w.l.o.g.
- Furthermore by a scaling argument it is sufficient to consider "z close to the origin", say $|z| < \rho^4$.
- Now $\boxed{\text{choose } k \ge 1}$ s.t. $\boxed{\rho^{k+3} < |z| < \rho^{k+2}}$ and note that

$$\begin{aligned} |D^2u(z) - D^2u(0)| &\le |D^2u(z) - D^2u_k(z)| \\ &\quad + |D^2u_k(z) - D^2u_k(0)| \\ &\quad + |D^2u_k(0) - D^2u(0)| \\ &= I_1 + I_2 + I_3. \end{aligned}$$

From Claim 2 and (3) we observe,

$$I_3 = |D^2 u_k(0) - \lim_{j\to\infty} D^2 u_j(0)|$$
$$\le \sum_{j=k}^{\infty} |D^2 u_j(0) - D^2 u_{j+1}(0)|$$
$$\le C \sum_{j=k}^{\infty} \omega(\rho^j) \le C \int_0^{|z|} \frac{\omega(r)}{r}\,dr$$

NB: The last estimate is done by bounding

$$\sum_{j=k}^{\infty} f(j) \le \int_k^{\infty} f(j-1)\,dj = \int_{k-1}^{\infty} f(\ell)\,d\ell$$

Thus
$$\sum_{j=k}^{\infty} \omega(\rho^j) \le \frac{1}{\log 2} \int_0^{\rho^{k-1}} \frac{\omega(y)}{y}\,dy$$

and
$$\rho^{k-1} \le \rho^4 |z| \le |z|, \text{ where (i) } \rho^{k+3} < |z| < \rho^{k+2}$$
(by the choice of k)
(ii) $\rho^4 |z| < 1$
(by assumption)

▶ We can also do a similar estimate for the term I_1 by looking at solutions to

$$\left.\begin{aligned} \Delta v &= f(z) \quad \text{in } B_j(z) \\ v &= u \quad \text{on } \partial B_j(z) \end{aligned}\right\} \quad j = k, k+1, \ldots$$

(i.e. translate from $x=0$ to $x=z$).

To estimate I_2, let $h_j = u_j - u_{j-1}$ for $j = 1, \ldots, k$.

As $D^2 h_j(z) - D^2 h_j(0) = D^3 h_j(\lambda z)\cdot z$ for some $0 \le \lambda \le 1$, then (2a) and (2b) imply that

$$|D^2 h_j(z) - D^2 h_j(0)| \le |z|\, \|D^3 h_j\|_{L^\infty(B_{j+2})}$$
$$\le C|z| \rho^{-j} \omega(\rho^j)$$
(since $|z| < \rho^{k+2}$)

Thus we have that

$$I_2 \le |D^2 h_k(z) - D^2 h_k(0)| + |D^2 u_{k-1}(z) - D^2 u_{k-1}(0)|$$
$$\le \sum_{j=1}^{k} |D^2 h_j(z) - D^2 h_j(0)| + |D^2 u_0(z) - D^2 u_0(0)|$$
$$\le |D^2 u_0(z) - D^2 u_0(0)| + C|z| \sum_{j=1}^{k} \rho^{-j} \omega(\rho^j)$$
$$\le |D^2 u_0(z) - D^2 u_0(0)| + C|z| \int_{|z|}^{1} \frac{\omega(r)}{r^2}\,dr .$$

To get a bound on the first term, note that

$$v_0 = u_0 - \frac{|x|^2}{2n} f(0) \quad \text{is harmonic in } B_1(0)$$

so that

$$|D^2 u_0(z) - D^2 u_0(0)| \le |z|\,|D^3 v_0(\lambda z)|, \text{ for some } 0<\lambda<1$$

and

$$|D^3 v_0(\lambda z)| \le \frac{C}{d^3} \sup |v_0| \le C\left(\sup_{B_1} |u_0| + |f(0)|\right)$$

since $d = \operatorname{dist}(\lambda z, \partial B_1) = 1 - |\lambda z| \ge 1 - \lambda\rho^{k+2} \ge 1 - \lambda\rho^2$.

To bound $|f(0)|$ in terms of u, we first assume w.l.o.g. that $f(0) > 0$.

Then consider $\quad v_0^\varepsilon = u_0 - \frac{|x|^2}{2n}(f(0) - \varepsilon)$

(NB: $\Delta v_0^\varepsilon = \varepsilon > 0$ for $\varepsilon > 0$)

Then the maximum principle implies

$$u_0(0) \leq \sup_{B_1} v_0^\varepsilon = \sup_{\partial B_1} u_0 - \frac{1}{2n}(f(0) - \varepsilon)$$

so that $\quad |f(0)| \leq C \sup_{B_1} |u_0|$.

Therefore,

$$I_2 \leq C|z| \left(\|w\|_{L^\infty(B_1)} + \int_{|z|}^1 \frac{\omega(r)}{r^2} dr \right)$$

as desired. ∎

SCHAUDER ESTIMATES FOR ELLIPTIC AND PARABOLIC EQUATIONS

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Introduction

The Schauder estimate for the Laplace equation was traditionally built upon the Newton potential theory. Different proofs were found later by Campanato [Ca], in which he introduced the Campanato space; Peetre [P], who used the convolution of functions; Trudinger [T], who used the mollification of functions; and Simon [Si], who used a blow-up argument. Also a perturbation argument was found by Safonov [S1,S2] and Caffarelli [C1, CC] for fully nonlinear uniformly elliptic equations, which also applies to the Laplace equation.

In this note we give an elementary and simple proof for the Schauder estimates for elliptic and parabolic equations. Our proof allows the right hand side to be Dini continuous and also give a sharp estimate for the modulus of continuity of the second derivatives. It also yields the log-Lipschitz continuity of the gradient for equations with bounded right hand side. Moreover, it also applies to nonlinear equations.

1. The Laplace equation

Consider the Laplace equation

$$\Delta u = f \quad \text{in } B_1(0), \tag{1.1}$$

where $B_1(0)$ is the unit ball in the Euclidean space $\mathbb{R}^n$. Suppose f is Dini continuous, namely $\int_0^1 \frac{\omega(r)}{r}dr < \infty$, where $\omega(r) = \sup_{|x-y|<r}|f(x)-f(y)|$. Then we have the following estimate for the modulus of continuity of D^2u.

Theorem 1. *Let $u \in C^2$ be a solution of (1.1). Then $\forall\ x, y \in B_{1/2}(0)$,*

$$|D^2u(x) - D^2u(y)| \le C_n\big[d \sup_{B_1}|u| + \int_0^d \frac{\omega(r)}{r} + d\int_d^1 \frac{\omega(r)}{r^2}\big], \tag{1.2}$$

where $d = |x-y|$, $C_n > 0$ depends only on n. It follows that if $f \in C^\alpha(B_1)$, then

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C_n\big[\sup_{B_1}|u| + \frac{\|f\|_{C^\alpha(B_1)}}{\alpha(1-\alpha)}\big] \quad \text{if } \alpha \in (0,1), \tag{1.3}$$

$$|D^2u(x) - D^2u(y)| \le C_n d\,(\sup_{B_1}|u| + \|f\|_{C^{0,1}}|\log d|\,) \quad \text{if } \alpha = 1. \tag{1.4}$$

* Work supported by the Australian Research Council.

Proof. We will use the following elementary estimates for harmonic functions,

$$|D^k w(0)| \le C_{n,k} r^{-|k|} \sup_{B_r}|w|, \tag{1.5}$$

where $C_{n,k}$ depends only on n and k. Simple proofs of (1.5) can be found in [E2].

Denote $B_k = B_{\rho^k}(0)$ $(\rho = \frac{1}{2})$. For $k = 0, 1, \cdots$, let u_k be the solution of

$$\Delta u_k = f(0) \quad \text{in } B_k, \qquad u_k = u \text{ on } \partial B_k.$$

Then $\Delta(u_k - u) = f(0) - f$. By the maximum principle,

$$\|u_k - u\|_{L^\infty(B_k)} \le C\rho^{2k}\omega(\rho^k). \tag{1.6}$$

Hence

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \le C\rho^{2k}\omega(\rho^k). \tag{1.7}$$

Since $u_{k+1} - u_k$ is harmonic, by (1.5) we have

$$\begin{aligned} &\|D(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \le C\rho^k\omega(\rho^k),\\ &\|D^2(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \le C\omega(\rho^k). \end{aligned} \tag{1.8}$$

Since $u \in C^2$, by (1.6), u_k minus the quadratic part of u is harmonic and is equal to $o(\rho^{2k})$ in B_k. Hence by (1.5),

$$\begin{aligned} Du(0) &= \lim_{k\to\infty} Du_k(0),\\ D^2u(0) &= \lim_{k\to\infty} D^2u_k(0). \end{aligned} \tag{1.9}$$

For any given point z near the origin, we have

$$\begin{aligned} &|D^2u(z) - D^2u(0)| \le I_1 + I_2 + I_3 =: \\ &|D^2u_k(z) - D^2u_k(0)| + |D^2u_k(0) - D^2u(0)| + |D^2u(z) - D^2u_k(z)|. \end{aligned} \tag{1.10}$$

Let $k \ge 1$ such that $\rho^{k+4} \le |z| \le \rho^{k+3}$. Then by (1.8), we have

$$I_2 \le C\sum_{j=k}^{\infty}\omega(\rho^k) \le C\int_0^{|z|}\frac{\omega(r)}{r}. \tag{1.11}$$

Similarly we can estimate I_3, through the solutions of $\Delta v = f(z)$ in $B_j(z)$ and $v = u$ on $\partial B_j(z)$ for $j = k, k+1, \cdots$. To estimate I_1, denote $h_j = u_j - u_{j-1}$. By (1.5) and (1.7) we have

$$|D^2h_j(z) - D^2h_j(0)| \le C\rho^{-j}\omega(\rho^j)|z|. \tag{1.12}$$

Hence

$$\begin{aligned} I_1 &\le |D^2u_{k-1}(z) - D^2u_{k-1}(0)| + |D^2h_k(z) - D^2h_k(0)|\\ &\le |D^2u_0(z) - D^2u_0(0)| + \textstyle\sum_{j=1}^k|D^2h_j(z) - D^2h_j(0)|\\ &\le C|z|(\|u_0\|_{L^\infty} + C\textstyle\sum\rho^{-j}\omega(\rho^j))\\ &\le C|z|(\|u\|_{L^\infty} + C\displaystyle\int_{|z|}^1\frac{\omega(r)}{r^2}). \end{aligned} \tag{1.13}$$

Combining (1.10), (1.11), and (1.13) we obtain (1.2). This completes the proof. □