SOBOLEV SPACES: INTRODUCTION

§ **Motivation.** If $u \in C^2(\Omega)$ is a solution to $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ — (A)

then

$$\int_\Omega Du \cdot D\varphi \, dx = \int_\Omega (-\Delta u)\varphi \, dx = \int_\Omega f\varphi \, dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

→ The bilinear form $\langle u, \varphi \rangle = \int_\Omega Du \cdot D\varphi \, dx$ is an inner product on $C_c^\infty(\Omega)$.

→ The completion $\mathcal{H} = \overline{C_c^\infty(\Omega)}^{\langle \cdot, \cdot \rangle}$ is a Hilbert space.

→ For "appropriate f", the linear functional

$$F : \varphi \mapsto \int_\Omega f\varphi \, dx$$

could be extended to a bounded linear functional on $\mathcal{H}$, i.e. $F \in \mathcal{H}'$!

→ Then by the Riesz representation theorem, $\exists u \in \mathcal{H}$ s.t.

$$\exists u \in \mathcal{H} \text{ s.t. } \langle u, \varphi \rangle = F(\varphi), \quad \forall \varphi \in C_c^\infty(\Omega).$$

This would then give the existence of a "generalised solution" to (A)

Remark. Thus classical existence questions are transformed into questions of regularity for generalised solutions.

§ **L^p spaces.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

For $1 \le p < \infty$, $L^p(\Omega)$ is the Banach space of measurable functions that are p-integrable with the norm

$$\|u\|_{p;\Omega} = \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{1/p}$$

and for $p = \infty$, $L^\infty(\Omega)$ denotes the space of bounded measurable functions with

$$\|u\|_{\infty;\Omega} = \operatorname*{ess\,sup}_\Omega |u| = \inf \{ k \ge 0 : |u(x)| \le k \text{ a.e. } x \in \Omega \}.$$

Remarks
(1) $L^p(\Omega)$ is separable for $p < \infty$.
(2) $L^p(\Omega)$ is reflexive for $1 < p < \infty$.
(3) $C^0(\bar{\Omega})$ is dense subset of $L^p(\Omega)$
(4) $L^p_{loc}(\Omega)$ = set of functions belonging to $L^p(\Omega')$ for every open $\Omega' \subset\subset \Omega$.

Proposition

(1) If $1 \le p \le \infty$ and $u_j \to u$ in L^p, then
$$\|u\|_{L^p} = \lim_{j \to \infty} \|u_j\|_{L^p}.$$
(2) If $1 < p < \infty$ and $u_j \rightharpoonup u$ in L^p, then there exists $k > 0$ s.t.
$$\|u_j\|_{L^p} \le k \quad \text{and} \quad \|u\|_{L^p} \le \liminf_{j \to \infty} \|u_j\|_{L^p}.$$
(NB: the result remains valid if $p = \infty$ and $u_j \overset{*}{\rightharpoonup} u$ in L^∞)

Remark. Weak convergence ensures the lower semicontinuity of the norm, while strong convergence guarantees its continuity.

recall the weak compactness of L^p:

<u>Lemma</u>. If $1<p<\infty$ and if $\exists k>0$ s.t. $\|u_j\|_{L^p} \le k$, then $\exists (u_{j_k})$ and $u \in L^p$ s.t. $u_{j_k} \rightharpoonup u$ in L^p.

Remark. The Bolzano–Weierstrass theorem(†) ascertains that from any bounded sequence one can extract a convergent subsequence. This is false in L^p but is true if the strong convergence is replaced by weak convergence.

→ <u>Young Inequality</u>: For $\frac{1}{p}+\frac{1}{q}=1$ and $a,b>0$ we have

$$ab \le \frac{a^p}{p}+\frac{b^q}{q}$$

(NB: $ab=(\varepsilon^{1/p}a)(\varepsilon^{-1/p}b) \le \varepsilon a^p + \varepsilon^{-q/p} b^q$)

→ <u>Hölder's Inequality</u>: For $\frac{1}{p_1}+\frac{1}{p_2}+\dots+\frac{1}{p_k}=1$ we have

$$\int_\Omega u_1 \cdots u_k \,dx \le \|u_1\|_{p_1} \cdots \|u_k\|_{p_k} \quad , \quad u_j \in L^{p_j}(\Omega).$$

As a consequence:

$$\left(\fint_\Omega u^p\right)^{1/p} \le \left(\fint_\Omega u^q\right)^{1/q} \quad \text{for } p\le q.$$

and

$$\|u\|_q \le \|u\|_p^{\lambda}\|u\|_r^{1-\lambda} \quad \text{for } p\le q\le r \text{ s.t.}$$

$$\frac{1}{q}=\frac{\lambda}{p}+\frac{1-\lambda}{r}.$$

(†) i.e. every <u>bounded</u> set in $\mathbb{R}^n$ is precompact
(its $\overset{\shortparallel}{\text{closure}}$ is compact)

§ Weak derivatives

<u>Def.</u> Let $u \in L^1_{loc}(\Omega)$ and α = multi-index. Then $v \in L^1_{loc}(\Omega)$ is called the <u>α-th weak derivative of u</u> if it satisfies

$$\int_\Omega \varphi v \,dx = (-1)^{|\alpha|}\int_\Omega u\, D^\alpha \varphi \,dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

We write $\boxed{v = D^\alpha u}$ and is uniquely determined up to a set of measure zero.

We call a function k-times weakly differentiable if all its weak derivatives of order k up to and including k exist.

<u>Def.</u> Let $W^k(\Omega)$ denote the linear space of <u>k-times weakly differentiable functions</u>.

NB: $C^k(\Omega) \subset W^k(\Omega)$, i.e. weakly diff. functions are a larger class.

<u>Lemma</u>. Let $u \in L^1_{loc}(\Omega)$. If $D^\alpha u$ exists for some multi-index α, then we have

$$D^\alpha u_\varepsilon(x) = (D^\alpha u)_\varepsilon(x)$$

for $\operatorname{dist}(x,\partial\Omega)>\varepsilon$.

Proof.

$$D^\alpha u_\varepsilon(x) = D^\alpha \int_\Omega \eta_\varepsilon(x-y)\, u(y)\, dy$$

$$= \int_\Omega D_x^\alpha \eta_\varepsilon(x-y)\, u(y)\, dy \quad \leftarrow \text{(diff. under the integral thm.)}$$

$$= (-1)^{|\alpha|} \int_\Omega D_y^\alpha \eta_\varepsilon(x-y)\, u(y)\, dy$$

$$= \int_\Omega \eta_\varepsilon(x-y)\, D^\alpha u(y)\, dy$$

$$= (D^\alpha u)_\varepsilon(x). \quad \blacksquare$$

Corollary If $u \in L^1_{loc}(\Omega)$ and there exists $D^\alpha u \in L^1_{loc}(\Omega)$ for some multi-index, α, then

$$D^\alpha u_\varepsilon \to D^\alpha u \quad \text{in } L^1_{loc} \quad \text{as } \varepsilon \to 0.$$

Proof. For $\Omega' \subset\subset \Omega$ we have

$$\| D^\alpha u_\varepsilon - D^\alpha u \|_{L^1(\Omega')} = \| (D^\alpha u)_\varepsilon - D^\alpha u \|_{L^1(\Omega')} \to 0 \quad \text{as } \varepsilon \to 0$$

by the properties of mollifiers (cf. Evans, Appendix, Thm. 7). $\blacksquare$

Theorem. If a given $u \in L^1_{loc}(\Omega)$ can be approximated by a sequence $(u_j) \subset C^\infty(\Omega)$ in the sense that

$$u_j \to u \quad \text{in } L^1_{loc}(\Omega)$$

and if, for $p > 1$ and any $\Omega' \subset\subset \Omega$, we have

$$\int_{\Omega'} |D^\alpha u_j|^p\, dx \le K, \qquad \text{— } (*)$$

where $K > 0$ is independent of j, then the α-th weak derivative $v = D^\alpha u$ of u exists and satisfies the condition

$$\int_{\Omega'} |D^\alpha u|^p\, dx \le K. \qquad \text{— } (**)$$

Proof of Thm. From $(*)$ and the L^p weak compactness lemma, there exists $(D^\alpha u_{j_k})$ s.t. $D^\alpha u_{j_k} \rightharpoonup v$, $v \in L^p$.

Now we have $\displaystyle\int_\Omega (u_{j_k} D^\alpha \varphi + (-1)^{|\alpha|+1} \varphi\, D^\alpha u_{j_k})\, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$

Passing to the limit $j_k \to \infty$ yields

$$\int_\Omega (u\, D^\alpha \varphi + (-1)^{|\alpha|+1} \varphi v)\, dx = 0$$

and $(**)$ follows by the above proposition for the lower-semicontinuity of $\|\cdot\|_{L^p}$. $\blacksquare$

Corollary. If the set of derivatives of α-th order of the mollified functions is weakly compact, then the given function has a weak derivative

§ The chain rule

Lemma. Let $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$ and $u \in W^1(\Omega)$.

Then $f \circ u \in W^1(\Omega)$ and $D(f \circ u) = f'(u) Du$.

Proof. Exercise.

The positive & negative parts of a function u are defined by

$$u^+ = \max(u, 0), \quad u^- = \min(u, 0) = -(-u)^+.$$

Then $u = u^+ + u^-$ and $|u| = u^+ - u^-$.

Lemma. Let $u \in W^1(\Omega)$. Then $u^+, u^-, |u| \in W^1(\Omega)$ and

$$Du^+ = \begin{cases} Du, & u > 0 \\ 0, & u \le 0, \end{cases} \quad Du^- = \begin{cases} 0, & u \ge 0 \\ Du, & u < 0, \end{cases} \quad D|u| = \begin{cases} Du, & u > 0 \\ 0, & u = 0 \\ -Du, & u < 0 \end{cases}.$$

Proof. Define

$$f_\varepsilon(u) = \begin{cases} \sqrt{u^2 + \varepsilon^2} - \varepsilon, & u > 0 \\ 0, & u \le 0 \end{cases}$$

By the previous lemma,

$$\int_\Omega f_\varepsilon(u)\, D\varphi \, dx = -\int_{u>0} \varphi \frac{u\, Du}{(u^2+\varepsilon^2)^{1/2}} \, dx$$

for any $\varphi \in C_c^\infty(\Omega)$. Then as $\varepsilon \to 0$ we have

$$\text{LHS} \to \int_\Omega u^+ D\varphi \, dx, \quad \text{RHS} \to -\int_{u>0} \varphi\, Du \, dx$$

which establishes the result for u^+. ∎