SOBOLEV SPACES

Def. For $k \in \mathbb{N}$ and $1 \le p \le \infty$ we let the Sobolev space

$$W^{k,p}(\Omega) = \{ u \in W^k(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \le k \}.$$

The space $W^{k,p}(\Omega)$ is clearly linear and has the norm

$$\|u\|_{k,p;\Omega} = \|u\|_{W^{k,p}(\Omega)} = \Big(\int_\Omega \sum_{|\alpha| \le k} |D^\alpha u|^p \, dx \Big)^{1/p}$$

or equivalently we could also use

$$\|u\|_{k,p;\Omega} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{p;\Omega}$$

Exercise: Show that $W^{k,p}(\Omega)$ is a Banach space.

Remark. The spaces $W^{k,p}(\Omega)$ are analogous to $C^{k,\alpha}(\bar\Omega)$ in the sense that:

$C^{k,\alpha}(\bar\Omega)$		$W^{k,p}(\Omega)$
continuous differentiability	$\longrightarrow$	weak derivatives
Hölder continuity	$\longrightarrow$	p-integrability

Def. Let $W_0^{k,p}(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ in the norm of $W^{k,p}(\Omega)$, i.e. $W_0^{k,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{k,p}}$.

Thus $u \in W_0^{k,p}(\Omega)$ iff $\exists u_j \in C_c^\infty(\Omega)$ s.t. $u_j \to u$ in the $\|\cdot\|_{k,p}$-norm

(i.e. functions $u \in W^{k,p}(\Omega)$ for which "$D^\alpha u = 0$ on $\partial\Omega$" $\forall |\alpha| \le k-1$).

NB: $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are separable for $1 \le p < \infty$ (and reflexive for $1 < p < \infty$).

Local spaces $W^{k,p}_{loc}(\Omega)$ can be defined as those functions belonging to $W^{k,p}(\Omega')$ for any $\Omega' \subset\subset \Omega$.

When $p = 2$, the spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ are Hilbert spaces under the inner product

$$(u,v)_k = \int_\Omega \sum_{|\alpha| \le k} D^\alpha u \, D^\alpha v \, dx .$$

Theorem (Properties of the weak derivative)

Let $u, v \in W^{k,p}(\Omega)$. Then we have:

(1) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multi-indices α, β s.t. $|\alpha| + |\beta| \le k$.

(2) $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ for all $|\alpha| \le k$ and $\lambda, \mu \in \mathbb{R}$.

(3) If $\Omega' \subset \Omega$ is an open subset of Ω, then $u \in W^{k,p}(\Omega')$.

(4) If $\psi \in C_c^\infty(\Omega)$, then $\psi u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\psi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \psi \, D^{\alpha-\beta} u$$

where $\binom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$.

Proof of (i): For any $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega D^\alpha u \, D^\beta \varphi \, dx = (-1)^{|\alpha|} \int_\Omega u \, D^{\alpha+\beta}\varphi \, dx$$

$$= (-1)^{|\alpha|} (-1)^{|\alpha+\beta|} \int_\Omega D^{\alpha+\beta} u \, \varphi \, dx$$

$$= (-1)^{|\beta|} \int_\Omega D^{\alpha+\beta} u \, \varphi \, dx$$

Thus $D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ in the weak sense. ∎

§ Density result

Recall that:

(1) if $u \in L^p_{loc}(\Omega)$, $p<\infty$, then

$$u_\varepsilon = \eta_\varepsilon * u \to u \text{ in } L^p_{loc}(\Omega).$$

(2) $(D^\alpha u)_\varepsilon(x) = (D^\alpha u_\varepsilon)(x)$, $x \in \Omega_\varepsilon$.

$\Rightarrow$ if $u \in W^{k,p}(\Omega)$ then $D^\alpha u_\varepsilon \to D^\alpha u$ in $L^p_{loc}(\Omega)$.

<u>Theorem</u>. The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Remark. For arbitrary Ω we <u>cannot</u> replace $C^\infty(\Omega)$ by $C^\infty(\bar\Omega)$ in this result.

<u>Def</u> (PoU) Let $\Omega \subset \mathbb{R}^n$ and $\{U_j\}_{j\in J}$ be an open cover of Ω. A <u>smooth partition of unity</u> (PoU) $\{\psi_j\}$ on Ω <u>subordinate to $\{U_j\}$</u> is a collection of smooth functions $\psi_j \in C^\infty(\Omega)$ s.t.

(1) $0 \le \psi_j \le 1$

(2) $\operatorname{supp}(\psi_j) \subset U_j$.

(3) $\forall x \in \Omega$ has open neighbourhood $V_x \subset \Omega$ s.t. $\operatorname{supp}\psi_j \cap V_x = \emptyset$ for all but finitely many $j \in J$.

(4) $\sum \psi_j = 1$ (NB: locally finite sum by (3)).

<u>Lemma</u> For any open cover $\{U_j\}$ of Ω, $\exists$ PoU $\{\psi_j\}$ subordinate to $\{U_j\}$.

i.e. "blend together" local data to get a global object without necessarily assuming agreement on overlaps.

Proof of Thm.

Let $\Omega_j = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\}$ be a collection of strictly contained subdomain of Ω s.t.

$$\Omega_j \subset\subset \Omega_{j+1} \quad \& \quad \bigcup_{j \geq 1} \Omega_j = \Omega.$$

Then let $\{\psi_j\}$ be a PoU subordinate to the cover $\{\Omega_{j+1} \setminus \bar{\Omega}_{j-1}\}_{j \in \mathbb{N}}$ where $\Omega_0 = \Omega_{-1} = \phi$ (the empty set).

NB: $\Omega_{j+1} \setminus \bar{\Omega}_{j-1} = \{x : \frac{1}{j+1} < \operatorname{dist}(x, \partial\Omega) < \frac{1}{j-1}\}$.

Now fix $u \in W^{1,p}(\Omega)$ and $\varepsilon > 0$. Then we can choose h_j s.t.

$$(\psi_j u)_{h_j} = \eta_{h_j} * u \in W^{1,p}(\Omega)$$

and

$$\left.\begin{aligned} & h_j < \operatorname{dist}(\Omega_j, \partial\Omega_{j+1}), \\ & \|(\psi_j u)_{h_j} - \psi_j u\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{2^j}. \end{aligned}\right\} \quad (*)$$

writing $v_j = (\psi_j u)_{h_j}$ we obtain from $(*)$ that only a finite number of v_j are non-vanishing on any given $\Omega' \subset\subset \Omega$.

Consequently $v = \sum v_j \in C^\infty(\Omega)$ and furthermore

$$\|u - v\|_{W^{1,p}(\Omega)} \leq \sum \|v_j - \psi_j u\|_{W^{1,p}(\Omega)} \leq \varepsilon,$$

since $u = \sum_{j \geq 1} \psi_j u$. $\blacksquare$

Proposition Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected.

If $u \in W^{1,p}(\Omega)$ s.t. $Du = 0$ a.e. (Ω), then $u = \text{const.}$ a.e. (Ω).

Proof. Let $\varepsilon > 0$. Consider $u_\varepsilon = \eta_\varepsilon * u \in C^\infty(\Omega_\varepsilon)$, where $\Omega_\varepsilon = \{x : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

Then

$$D_j u_\varepsilon = \eta_\varepsilon * D_j u, \quad j = 1, 2, \dots, n.$$

Thus $D_j u_\varepsilon = 0$ on Ω_ε by assumption.

Hence $u_\varepsilon = \text{const}$ on bounded subsets of Ω_ε.

Choose $x, y \in \Omega$. Then $\exists$ a curve $\Gamma \subset \Omega$ from x to y by connectedness.

Then $\Gamma \subset \Omega_\varepsilon$ and x, y are in some connected subset of Ω_ε. Hence $u_\varepsilon(x) = u_\varepsilon(y) = \text{const.}$ and as $u_\varepsilon \to u$ a.e. (Ω) we conclude $u = \text{const.}$ a.e. (Ω). $\blacksquare$