IMBEDDING RESULTS

Idea: want to sacrifice differentiability for improved integrability.

Recall that a Banach space B_1 is continuously embedded into a Banach space B_2 (notation $B_1 \to B_2$) if there is a bounded linear one-to-one mapping $B_1 \to B_2$.

Theorem I (Sobolev)

Let $\Omega \subset \mathbb{R}^n$ be open, then

$$W_0^{1,p}(\Omega) \to \begin{cases} L^{np/(n-p)}(\Omega), & \text{if } p<n \\ C^0(\bar{\Omega}), & \text{if } p>n \end{cases}.$$

Furthermore, there exists a constant $c = c(n,p) > 0$ s.t. for any $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{np/(n-p);\Omega} \le c \|Du\|_{p;\Omega} \quad \text{for } p<n$$

$$\operatorname*{ess\,sup}_{\Omega} |u| \le c |\Omega|^{\frac{1}{n}-\frac{1}{p}} \|Du\|_{p;\Omega} \quad \text{for } p>n$$

Corollary. For $k \in \mathbb{N}$, $u \in W_0^{1,p}(\Omega)$, $1 \le p \frac{n}{k}$ we have $W_0^{k,p}(\Omega) \to L^{\frac{np}{n-kp}}(\Omega)$ with

$$\|u\|_{\frac{np}{n-kp};\Omega} \le c(n,p,k) \|D^k u\|_{p;\Omega}.$$

Proof. By induction over k we have

$$W_0^{k,p}(\Omega) \hookrightarrow W^{k-1,\frac{np}{n-p}}(\Omega) \hookrightarrow W^{k-2,\frac{np}{n-2p}}(\Omega) \hookrightarrow \dots \hookrightarrow L^{\frac{np}{n-kp}}(\Omega)$$

because if $q = \frac{np}{n-jp}$ then $\frac{nq}{n-q} = \frac{np}{n-(j+1)p}$ for $j = 1,2,\dots,j-1$. ∎

Proof. We prove the 1st part by a method due Nirenberg (1959).

Firstly, consider $u \in C_c^\infty(\Omega)$ and $p=1$.

Extend u to $\mathbb{R}^n$ by setting $u=0$ on $\mathbb{R}^n \setminus \Omega$.

Then note that

$$u(x) = \int_{-\infty}^{x_i} D_i u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i$$

and so

$$|u(x)| \le \int_{-\infty}^{\infty} |D_i u|\, dy_i \quad \text{for } i = 1, \dots, n.$$

By taking the product of these inequalities with themselves we find

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |D_i u|\, dy_i \right)^{\frac{1}{n-1}}.$$

Integrating the latter w.r.t. x_1 yields

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |D_i u|\, dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |D_1 u|\, dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |D_i u|\, dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |D_1 u|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |D_i u|\, dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$\le \left(\int_{-\infty}^{\infty} |D_1 u|\, dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_i u|\, dy_i\, dx_1 \right]^{\frac{1}{n-1}} \quad \text{—} (*)$$

where the last inequality follows by Hölder's ineq, i.e.

$$\int u_2 \cdots u_n\, dx_1 \le \|u_2\|_{n-1} \cdots \|u_n\|_{n-1} \quad \text{with } u_i = \left(\int_{-\infty}^{\infty} |D_i u|\, dy_i \right)^{\frac{1}{n-1}}.$$

Now integrate (*) w.r.t. x_2 : This yields

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1\right) dx_2$$

$$\leq \int_{-\infty}^{\infty}\Big[\Big(\int_{-\infty}^{\infty} |D_1 u| dy_1\Big)^{\frac{1}{n-1}} \underbrace{\prod_{i=2}^{n}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D_i u| dy_i dx_i\Big)^{\frac{1}{n-1}}}\Big] dx_2$$

$$= \Big[\iint |D_2 u| dy_2 dx_1\Big]^{\frac{1}{n-1}} \prod_{i=3}^{n}\Big[\iint |D_i u| dy_i dx_1\Big]^{\frac{1}{n-1}}$$

$$= \Big[\iint |D_2 u| dy_2 dx_1\Big]^{\frac{1}{n-1}} \int dx_2 \Big\{\Big(\int |D_1 u| dy_1\Big)^{\frac{1}{n-1}} \prod_{i=3}^{n}\Big(\iint |D_i u| dy_i dx_1\Big)^{\frac{1}{n-1}}\Big\}$$

$$\leq \Big(\iint |D_2 u| dy_2 dx_1\Big)^{\frac{1}{n-1}} \Big(\iint |D_1 u| dx_2 dy_1\Big)^{\frac{1}{n-1}} \Big(\prod_{i=3}^{n}\iiint |D_i u| dx_1 dx_2 dy_i\Big)^{\frac{1}{n-1}}.$$

Continuing in this way yields

$$\int_{\mathbb{R}^\ell} |u(x)|^{\frac{n}{n-1}} dx_1 \dots dx_\ell$$

$$\leq \prod_{j=1}^{\ell}\Big(\int_{\mathbb{R}^\ell} |D_j u| dy_1 \dots dy_\ell\Big)^{\frac{1}{n-1}} \prod_{j=\ell+1}^{n}\Big(\int_{\mathbb{R}^{\ell+1}} |D_j u| dy_1 \dots dy_\ell dy_j\Big)^{\frac{1}{n-1}}$$

Then when $\ell = n$ we find that

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq \prod_{j=1}^{n}\Big(\int_{\mathbb{R}^n} |D_j u| dy_1 \dots dy_n\Big)^{\frac{1}{n-1}}$$

$$\leq \Big(\int_{\mathbb{R}^n} |Du| dx\Big)^{\frac{n}{n-1}}. \quad \text{—— (**)}$$

This yields the desired estimate for $p = 1$.

Now consider the case $1 < p < n$:

Apply the estimate (**) to $v = |u|^\gamma$, $\gamma > 1$.

This gives

$$\| |u|^\gamma \|_{L^{n/(n-1)}} \leq \| D|u|^\gamma \|_{L^1} = \gamma \int |u|^{\gamma-1} |Du| dx$$

$$\leq \gamma \Big(\int |u|^{(\gamma-1)\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \Big(\int |Du|^p\Big)^{1/p}.$$

Now choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$, i.e. $\gamma = \frac{p(n-1)}{n-p} > 1$.

Consequently we obtain (NB: $p < n$)

$$\boxed{\|u\|_{\frac{np}{n-p}} \leq \gamma \|Du\|_p.} \quad \text{—— (+)}$$

To extend the estimate for arbitrary $u \in W_0^{1,p}(\Omega)$, let (u_j) be a sequence of $C_c^\infty(\Omega)$ - functions s.t. $u_j \to u$ in $W^{1,p}$-norm. Apply the estimate (+) to u_j and the difference $u_j - u_k$ gives

$$\|u_j\|_{\frac{np}{n-p}} \leq \gamma \|Du_j\|_p \quad \& \quad \|u_j - u_k\|_{\frac{np}{n-p}} \leq \gamma \|Du_j - Du_k\|_p.$$

Hence $\{u_j\}$ is Cauchy sequence in $L^{\frac{np}{n-p}}(\Omega)$.

Consequently the limit u will be in $L^{\frac{np}{n-p}}(\Omega)$ and satisfy an inequality of the form $\|u\|_{\frac{np}{n-p}} \leq c\|Du\|_p$. ∎

By iterating the previous theorem we obtain the following more general version.

<u>Theorem II (Sobolev)</u>

Let $\Omega \subset \mathbb{R}^n$ be open. Then for each $k \in \mathbb{N}$ we have

$$W_0^{k,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega) & \text{for } 1 \le q \le \dfrac{1}{\frac{1}{p} - \frac{k}{n}} & \text{if } 1 \le p < \frac{n}{k} \\ L^{q^*}(\Omega) & \text{for } 1 \le q^* < \infty & \text{if } 1 < p = \frac{n}{k} \\ C^m(\bar{\Omega}) & \text{for } 0 \le m \le k - \frac{n}{p} & \text{if } p > \frac{n}{k} \end{cases}$$

NB: $C^{k-\frac{n}{p}}(\bar{\Omega}) = C^{\ell, k-\frac{n}{p}-\ell}(\bar{\Omega})$ for $\ell \in \mathbb{N}_0$ s.t. $0 \le \ell < k - \frac{n}{p} < \ell + 1$.

<u>Remark</u>. In general $W_0^{k,p}(\Omega)$ <u>cannot</u> be replaced by $W^{k,p}(\Omega)$ in this theorem.

However this replacement can be done for "nice" domains. In particular, for domains with Lipschitz continuous boundary, we have

$$W^{k,p}(\Omega) \hookrightarrow \begin{cases} L^{np/(n-kp)}(\Omega) & \text{if } 1 \le p < \frac{n}{k} \\ C_B^{k-\frac{n}{p}}(\Omega) & \text{if } p > \frac{n}{k} \end{cases}$$

where $C_B^{k-\frac{n}{p}}(\Omega) = \{ u \in C^{\lfloor k-\frac{n}{p} \rfloor, k-\frac{n}{p}-\lfloor k-\frac{n}{p} \rfloor}(\Omega) : D^\alpha u \in L^\infty(\Omega), |\alpha| \le \lfloor k - \frac{n}{p} \rfloor \}$

i.e. the derivatives of order $\lfloor k - \frac{n}{p} \rfloor$ are Hölder continuous in the L^∞-metric. In other words, for any multi-index α s.t. $0 < k - \frac{n}{p} - |\alpha| < 1$ the derivatives $D^\alpha u$ satisfy a Hölder condition in the L^∞-metric of the form

$$\sup_{x \in \Omega} |D^\alpha u(x+h) - D^\alpha u(x)| \le C \|u\|_{W^{k,p}(\Omega)} |h|^{k - \frac{n}{p} - |\alpha|}.$$