EXTENSION & TRACE RESULTS

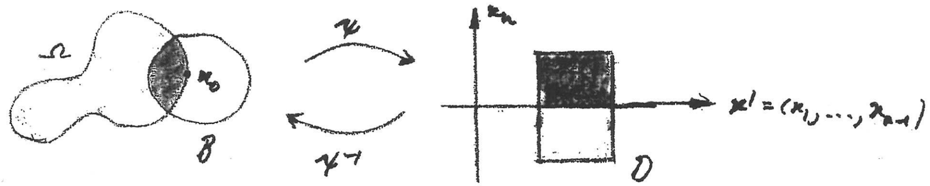
§ Boundary Values

Def. A bounded domain $\Omega \subset \mathbb{R}^n$, and its boundary $\partial\Omega$, is said to be of class $C^{k,\alpha}$ if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a one to one map ψ from B onto $D \subset \mathbb{R}^n$ s.t.

(i) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$

(ii) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$

(iii) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$



e.g. if $\psi, \psi^{-1} \in C^{0,1}$ we say Ω is a Lipschitz domain.

Remark. A domain Ω is said to have a boundary portion $T \subset \partial\Omega$ of class $C^{k,\alpha}$ if at each point $x_0 \in T$ there is a ball $B = B(x_0)$ in which the above conditions are satisfied with $B \cap \partial\Omega \subset T$.

In such a situation we shall say ψ "rectifies" or "straighten out" the boundary near x_0.

§ Extension problem

On any bounded domain $\Omega \subset \mathbb{R}^n$, the extension of any $W^{1,p}$-function by zero (i.e. by setting $u = 0$ on $\mathbb{R}^n \setminus \Omega$) could create a discontinuity along Ω (in the sense that the extended function $\tilde{u}$ on $\mathbb{R}^n$ might no longer have a derivative on $\mathbb{R}^n$ and/or might not be in $W^{1,p}(\mathbb{R}^n)$). Hence we need to construct an extension $\tilde{u}$ s.t. the derivatives $\frac{\partial \tilde{u}}{\partial x_i}$ exist on the extended domain $\mathbb{R}^n$ and that $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$, i.e. the extension operator E is well defined and bounded.

Notation. Let

$$\mathbb{R}^n_+ = \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0 \}$$

$$Q = \{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1, |x_n| < 1 \}$$

$$Q_+ = Q \cap \mathbb{R}^n_+$$

$$Q_0 = \{ x = (x', 0) : |x'| < 1 \}$$

where $|x'| = (x_1^2 + x_2^2 + \dots + x_{n-1}^2)^{1/2}$.

§ The Extension of functions in the class $W^{k,p}(\Omega)$

A method of successive reflection by Whitney & Hestenes allows one to extend functions from a sufficiently smooth bounded domain into a larger one that remains in the same $C^{k,\alpha}$-regularity class. Namely, we have

<u>Lemma</u> (GT, p.136)

Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,\alpha}$-domain, and let Ω' be an open set containing $\bar{\Omega}$. Suppose $u \in C^{k,\alpha}(\bar{\Omega})$, then there exists $w \in C^{k,\alpha}_c(\Omega')$ s.t.

$$w = u \text{ in } \Omega$$

and

$$\|w\|_{C^{k,\alpha}(\Omega')} \le c(k,\Omega,\Omega')\|u\|_{C^{k,\alpha}(\Omega)}.$$

<u>Remark.</u> Babich (1953) showed this method extends to the class $W^{k,p}(\Omega)$.

<u>Theorem</u> (Extension)

Let $\Omega \subset \mathbb{R}^n$ be a $C^{k-1,1}$-domain, $1 \le p < \infty$ and $k \in \mathbb{N}$. Then

for any open set $\Omega' \supset\supset \Omega$ there exists a bounded linear extension operator

$$E : W^{k,p}(\Omega) \to W^{k,p}_0(\Omega')$$

s.t. $$Eu = u \text{ in } \Omega$$

and $$\|Eu\|_{k,p;\Omega'} \le c(k,\Omega,\Omega')\|u\|_{k,p;\Omega}$$

for all $u \in W^{k,p}(\Omega)$.

<u>Corollary 1.</u> Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $1 \le p < \infty$. Then the subspace $C^\infty(\bar{\Omega}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.

<u>Proof.</u> Given $u \in W^{1,p}(\Omega)$ we have $v = Eu \in W^{1,p}(\mathbb{R}^n)$. Then by the density theorem applied on $\mathbb{R}^n$, there exists $(v_k) \subset C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ s.t. $v_k \to v$ in $W^{1,p}(\mathbb{R}^n)$. In which case $u_k = v_k|_\Omega \in C^\infty(\bar{\Omega})$ and $u_k \to u$ in $W^{1,p}(\Omega)$. ∎

<u>Corollary 2.</u> Let $\Omega \subset \mathbb{R}^n$ be a $C^{0,1}$-domain.

For $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$, there exists $(u_j) \subset C^\infty_c(\mathbb{R}^n)$ s.t.

$$u_j|_\Omega \to u \text{ in } W^{1,p}(\Omega).$$

(i.e. the restriction of $C^\infty_c(\mathbb{R}^n)$ forms a dense subspace of $W^{1,p}(\Omega)$)

Theorem A

Let $u \in W^{k,p}(Q+)$, $1 \le p < \infty$, and $k \in \mathbb{N}$ then there exists an Extension operator

$$E_0 : W^{k,p}(Q_+) \to W^{k,p}(Q)$$

s.t. $E_0 u = u$ in Ω

with

$$\| E_0 u \|_{W^{k,p}(Q)} \le C(n,k) \| u \|_{W^{k,p}(Q+)} .$$

Rmk: If $k = 1$ then

$$E_0 u(x', x_n) = \begin{cases} u(x', x_n), & x_n > 0 \\ u(x', -x_n), & x_n < 0 \end{cases}$$

and

$$\frac{\partial}{\partial x_i} E_0 u = \begin{cases} \frac{\partial u}{\partial x_i}(x', x_n), & x_n > 0 \\ \frac{\partial u}{\partial x_i}(x', -x_n), & x_n < 0 \end{cases},$$

$$\frac{\partial}{\partial x_n} E_0 u = \begin{cases} \frac{\partial u}{\partial x_n}(x', x_n), & x_n > 0 \\ -\frac{\partial u}{\partial x_n}(x', -x_n), & x_n < 0 \end{cases}.$$

Proof of Theorem A:

Given $u \in W^{k,p}(Q_+)$ we can define an extension $E_0 u$ of u to all of Q by setting

$$E_0 u(x) = \begin{cases} u(x), & x_n > 0 \\ \tilde{u}(x), & x_n < 0 \end{cases}$$

where

$$\tilde{u}(x) = \lambda_1 u(x', -x_n) + \lambda_2 u(x', -\tfrac{1}{2} x_n) + \dots + \lambda_k u(x', -\tfrac{1}{k} x_n)$$

and the constants $\lambda_1, \dots, \lambda_k$ are determined by the system of equations

$$\left.\begin{aligned} &(-1)^m \lambda_1 + \left(\tfrac{-1}{2}\right)^m \lambda_2 + \dots + \left(\tfrac{-1}{k}\right)^m \lambda_k = 1 \\ &\text{for } m = 0, 1, \dots, k-1. \end{aligned}\right\} \quad (*)$$

Now if $u \in C^\infty(Q_+) \cap W^{k,p}(Q_+)$ we need to check that the extended function exists across the boundary. (i.e. it has a derivative that exists)

To do this, consider $l_1, l_2 \in \mathbb{N}$ s.t. $l_1 + l_2 \le k$ and let $\varphi \in C_c^\infty(Q)$.

Then integration by parts l_1-times w.r.t. x_n yields (†)

$$\int_{Q_+} \varphi \frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} u \, dx = (-1)^{l_1} \int_{Q_+} \frac{\partial^{l_1}}{\partial x_n^{l_1}} \varphi \, D_{x'}^{l_2} u \, dx - \sum_{q=0}^{l_1-1} (-1)^q \int_{\partial Q_+} \frac{\partial^{l_1-q-1}}{\partial x_n^{l_1-q-1}} \varphi \frac{\partial^q}{\partial x_n^q} D_{x'}^{l_2} u \, dx'$$

and

$$\int_{Q_-} \varphi \frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} \hat{u} \, dx = (-1)^{l_1} \int_{Q_-} \frac{\partial^{l_1}}{\partial x_n^{l_1}} \varphi \, D_{x'}^{l_2} \hat{u} \, dx + \sum_{q=0}^{l_1-1} (-1)^q \int_{\partial Q_-} \frac{\partial^{l_1-q-1}}{\partial x_n^{l_1-q-1}} \varphi \frac{\partial^q}{\partial x_n^q} D_{x'}^{l_2} \hat{u} \, dx' .$$

From the definition of $\hat{u}$ we also have

$$\frac{\partial^q}{\partial x_n^q} D_{x'}^{l_2} \hat{u} \Big|_{x_n=0} = \frac{\partial^q}{\partial x_n^q} D_{x'}^{l_2} \left(\sum_{j=1}^k \lambda_j u\left(x', \tfrac{-1}{j} x_n\right) \right) \Big|_{x_n=0}$$

$$= \sum_{j=1}^k \lambda_j \left(-\frac{1}{j}\right)^q \left(\frac{\partial^q}{\partial x_n^q} D_{x'}^{l_2} u \right)(x', 0)$$

$$= \frac{\partial^q}{\partial x_n^q} D_{x'}^{l_2} u \Big|_{x_n=0} \quad \text{by } (*) .$$

(†) i.e. by applying the 1-dimensional formula

$$\int f^{(n+1)} \psi \, dx = \psi f^{(n)} - \psi' f^{(n-1)} + \dots + (-1)^n \psi^{(n)} f + (-1)^{n+1} \int \psi^{(n+1)} f \, dx .$$

It now follows that

$$\int_{Q_+} \varphi \frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} u \, dx + \int_{Q_-} \varphi \frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} \hat{u} \, dx$$

$$= (-1)^{l_1} \int_{Q_+} \frac{\partial^{l_1}}{\partial x_n^{l_1}} \varphi \, D_{x'}^{l_2} u \, dx + (-1)^{l_1} \int_{Q_+} \frac{\partial^{l_1}}{\partial x_n^{l_1}} \varphi \, D_{x'}^{l_2} \tilde{u} \, dx$$

$$= (-1)^{l_1+l_2} \int_Q \left(\frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} \varphi \right) (E_0 u) \, dx .$$

Therefore the weak derivatives

$$\frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} (E_0 u) = \begin{cases} \dfrac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} u , & x_n > 0 \\[2ex] \dfrac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} \tilde{u} , & x_n < 0 \end{cases}$$

exist.

It is also easy to see that

$$\left\| \frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} (E_0 u) \right\|_{L^p(Q)} \le c \left\| \frac{\partial^{l_1}}{\partial x_n^{l_1}} D_{x'}^{l_2} u \right\|_{L^p(Q_+)}$$

and so $E_0 u \in C^k(Q) \cap W^{k,p}(Q)$ with

$$\| E_0 u \|_{W^{k,p}(Q)} \le c(n,k) \| u \|_{W^{k,p}(Q_+)} .$$

Hence by the BLT theorem, we have a bounded linear operator $E_0 : W^{k,p}(Q_+) \to W^{k,p}(Q)$. ∎

1953 г. март—апрель *т. VIII, вып. 2 (54)*

УСПЕХИ МАТЕМАТИЧЕСКИХ НАУК

НАУЧНЫЕ СООБЩЕНИЯ И ЗАДАЧИ

К ВОПРОСУ О РАСПРОСТРАНЕНИИ ФУНКЦИЙ

В. М. Бабич

Пусть граница области такова, что для окрестности каждой точки $x'\{x_1', \ldots, x_n'\}$ границы существует преобразование

$$x_i = x_i(t_1, t_2, \ldots, t_n) \qquad (i = 1, 2, \ldots, n), \tag{*}$$

удовлетворяющее условиям:

1. $$x_i(t_1, t_2, \ldots, t_n) \in C^{(l)},$$
2. $$\frac{D(x_1, x_2, \ldots, x_n)}{D(t_1, t_2, \ldots, t_n)} \neq 0.$$

3. Точки границы в рассматриваемой окрестности точки x' являются образами множества точек $t\ \{t_1, t_2, \ldots, t_n\}$, определяемого соотношениями:

$$t_j \geqslant 0, \ t_1 t_2 \ldots t_k = 0, \ 1 \leqslant j \leqslant k \leqslant n.$$

Условимся называть такую границу кусочно-гладкой границей класса $C^{(l)}$.

Уитнеем и Хестинсом доказана следующая теорема.

Если функция $\varphi(x_1, x_2, \ldots, x_n) \in C^{(l)}$ в области Ω, ограниченной кусочно-гладким контуром класса $C^{(l)}$, то её можно распространить на всё пространство с сохранением класса (см. [1,2]; для $n=2$ доказательство имеется в [3]).

Доказательство это сохраняет в основном силу, если рассматривать вместо непрерывных обобщённые производные.

Точнее, имеет место теорема [1]):

Если функция $\varphi(x_1, x_2, \ldots, x_n) \in W_p^{(l)}$ в области, ограниченной кусочно-гладким контуром класса $C^{(l)}$, то её можно распространить на всё пространство так, что она будет принадлежать классу $W_p^{(l)}$ во всякой конечной области [2]).

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[1]) Как автору стало известно после сдачи работы в печать, аналогичный результат был получен С. М. Никольским и доложен им на конференции по дифференциальным уравнениям в мае 1952 г.

[2]) В обозначениях мы следуем монографии [4].

Докажем следующую лемму:

Пусть функция $f(x, y) \in W_p^{(l)}$ в области $\Omega \left\{ \begin{array}{l} a < x < b \\ 0 < y < \Delta \end{array} \right.$. Тогда существует распространение функции f с сохранением класса на весь прямоугольник $\Omega^* = (a, b, -\Delta, \Delta)$.

(2 переменных вместо n взяты лишь для упрощения записи).

Определим l чисел $\lambda_1, \lambda_2, \ldots, \lambda_l$ из системы уравнений

$$(-1)^k \lambda_1 + \left(-\frac{1}{2}\right)^k \lambda_2 + \ldots + \left(-\frac{1}{l}\right)^k \lambda_l = 1 \qquad (k = 0, 1, \ldots, l-1).$$

Пусть

$$\tilde{f}(x, y) = \lambda_1 f(x, -y) + \lambda_2 f\left(x, -\frac{1}{2} y\right) + \ldots + \lambda_l f\left(x, -\frac{1}{l} y\right).$$

Докажем, что f^*, равная f в Ω и $\tilde{f}$ в $\Omega^* - \Omega$, принадлежит $W_p^{(l)}$ в Ω^*.

Пусть ψ достаточно гладка в Ω^* и равна нулю в пограничной полоске. Составим интегралы:

$$J_1 = \iint_{\Omega} \psi \frac{\partial^l f}{\partial x^\alpha \partial y^{l-\alpha}} dx\,dy \quad \text{и} \quad J_2 = \iint_{\Omega^* - \Omega} \psi \frac{\partial^l \tilde{f}}{\partial x^\alpha \partial y^{l-\alpha}} dx\,dy.$$

Применяем l раз формулу Остроградского (её справедливость в этом случае легко доказывается):

$$J_1 = (-1)^l \iint_{\Omega} f \frac{\partial^l \psi}{\partial x^\alpha \partial y^{l-\alpha}} dx\,dy - \int_a^b \left[\frac{\partial^{l-1} f}{\partial x^\alpha \partial y^{l-\alpha-1}} \psi + \ldots + (-1)^{l-\alpha} \frac{\partial^\alpha f}{\partial x^\alpha} \cdot \frac{\partial^{l-\alpha-1} \psi}{\partial y^{l-\alpha-1}} \right] dx,$$

$$J_2 = (-1)^l \iint_{\Omega^* - \Omega} \tilde{f} \frac{\partial^l \psi}{\partial x^\alpha \partial y^{l-\alpha}} dx\,dy + \\ + \int_a^b \left[\frac{\partial^{l-1} \tilde{f}}{\partial x^\alpha \partial y^{l-\alpha-1}} \psi + \ldots + (-1)^{l-\alpha} \frac{\partial^\alpha \tilde{f}}{\partial x^\alpha} \cdot \frac{\partial^{l-\alpha-1} \psi}{\partial y^{l-\alpha-1}} \right] dx.$$

Если учесть равенства:

$$\left. \frac{\partial^{l-k} \tilde{f}}{\partial x^\alpha \partial y^{l-\alpha-k}} \right|_{y=0} = \left[(-1)^{l-\alpha-k} \lambda_1 + \ldots + \left(-\frac{1}{l}\right)^{l-\alpha-k} \lambda_l \right] \left. \frac{\partial^{l-k} f}{\partial x^\alpha \partial y^{l-\alpha-k}} \right|_{y=0} = \\ = \left. \frac{\partial^{l-k} f}{\partial x^\alpha \partial y^{l-\alpha-k}} \right|_{y=0},$$

то очевидно, что

$$J_1 + J_2 = \iint_{\Omega} \psi \frac{\partial^l f}{\partial x^\alpha \partial y^{l-\alpha}} dx\,dy + \iint_{\Omega^* - \Omega} \psi \frac{\partial^l \tilde{f}}{\partial x^\alpha \partial y^{l-\alpha}} dx\,dy = \\ = (-1)^l \iint_{\Omega^*} f^* \frac{\partial^l \psi}{\partial x^\alpha \partial y^{l-\alpha}} dx\,dy;$$

кроме того,

$$\iint_{\Omega} \left| \frac{\partial^l f}{\partial x^\alpha \partial y^{l-\alpha}} \right|^p dx\,dy + \iint_{\Omega^* - \Omega} \left| \frac{\partial^l \tilde{f}}{\partial x^\alpha \partial y^{l-\alpha}} \right|^p dx\,dy < +\infty,$$

так как $f \in W_p^{(l)}$ на Ω.

Две последние формулы говорят о том, что $f^* \in W_p^{(l)}$ на Ω^*.

Пусть $\varphi \in W_p^{(l)}$ в области, ограниченной кусочно-гладким контуром класса $C^{(l)}$.

Пользуясь доказанной леммой и преобразованием координат (*), легко показать, что существует искомое распространение φ с сохранением класса на малую окрестность любой точки границы. Как следует из рассуждений Хестенса (см. [2] или [3]), этого достаточно для того, чтобы φ можно было распространить на всё пространство с сохранением класса.

Пусть φ^*—такое распространение функции φ. Последовательность φ_h^* средних функций для φ^* [1]) является последовательностью функций класса $C^{(l)}$ в $\overline{\Omega}$, сходящихся к φ в Ω в смысле метрики $W_p^{(l)}$.

Для всякой области φ_h^* можно как угодно точно аппроксимировать в смысле метрики $W_p^{(l)}$ многочленом с рациональными коэффициентами.

Следствием этого является сепарабельность пространства $W_p^{(l)}$ для областей с кусочно-гладкой границей класса $C^{(l)}$.

Все эти простые замечания часто бывают полезны во многих теоретических вопросах математической физики.

Поступило в редакцию 1 июля 1952 г.

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

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[1]) О средних функциях см. [4].

Babič, V. M.: Zur Frage der Fortsetzung von Funktionen. Uspechi mat. Nauk 8, Nr. 2 (54), 111—113 (1953) [Russisch].

Insbesondere Whitney (dies. Zbl. 8, 249) und später Hestenes (dies. Zbl. 24, 386) haben die Frage nach der Fortsetzbarkeit differenzierbarer Funktionen mehrerer Veränderlicher untersucht. Verf. überträgt das Hauptergebnis von Whitney für die Funktionen der Klasse C^l (vgl. dies. Zbl. 24, 386) auf die Funktionen der Klasse W_p^l, deren sämtliche Ableitungen l-ter Ordnung zur Klasse L^p gehören. Umfassendere Resultate finden sich bei Nikolskij (dies. Zbl. 46, 61); vgl. auch das folgende Referat. *L. Schmetterer.*

MR0056675 (15,110f) 27.2X
Babič, V. M. [Babich, Vasiliĭ Mikhaĭlovich]
On the extension of functions. (Russian)
Uspehi Matem. Nauk (N.S.) 8, (1953). *no. 2(54)*, 111–113

Let A be a closed bounded region in E_n whose boundary is piecewise of class $C^{(k)}$, and let $f(p)$, $p \in A$, be a function defined in A and of class $C^{(k)}$ in A, i.e., f is continuous with all its partial derivatives of order k in A. H. Whitney [Trans. Amer. Math. Soc. **36**, 63–89 (1934)] and M. R. Hestenes [Duke Math. J. 8, 183–192 (1941); MR0003434] have proved various theorems on the extension of functions, in particular that every function of class $C^{(k)}$ in A has an extension which is of class $C^{(k)}$ in E_n. The author says that a function f is of class $W_p{}^{(k)}$ if the continuity requirement of the partial derivatives of order k in the definition of class $C^{(k)}$ is replaced by L^p-integrability. Then he proves that every function f of class $W_p{}^{(k)}$ in A has an extension which is of class $W_p{}^{(k)}$ in E_n. Hestenes' method based on a modified form of reflection principle is used. *L. Cesari*

[Slo58b] ———, *Generalized Sobolev spaces and their application to boundary problems for partial differential equations*, Leningrad. Gos. Ped. Inst. Učen. Zap. **197** (1958), 54–112. (English transl. in Amer. Math. Soc. Transl. (2) **57** (1966), 207–275).

GENERALIZED SOBOLEV SPACES AND THEIR APPLICATION TO BOUNDARY PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

L. N. SLOBODECKIĬ

The behavior of the values of various classes of functions of several independent variables on manifolds of various dimensions is of great significance in the theory of boundary problems for partial differential equations. Numerous investigations, both of Soviet and foreign authors, have been devoted to this question. The present article is also related to this question and it gives an account of the results which the author obtained at the beginning of 1957.

Whilst preparing this article for printing the author profited from the valuable advice of Professors M. I. Višik, O. A. Ladyženskaja and H. L. Smolickiĭ. The author takes this opportunity to express his heart felt thanks to them.

The author expresses his sincere gratitude also to Professor S. G. Mihlin whc undertook the big task of editing this report.

Introduction

Let E_n denote the n-dimensional euclidean space of points $x = (x_1, \cdots, x_n)$ let $n_1, \cdots, n_r$ be natural numbers whose sum is equal to n and let $E^{(k)}$ denote n_k-dimensional spaces of points $x^{(k)} = (x_1^{(k)}, \cdots, x_{n_k}^{(k)})$ $(k = 1, 2, \cdots, r)$. Suppose further that $\Omega^{(k)}$ is a finite or infinite domain of the space $E^{(k)}$ $(k = 1, 2, \cdots,$ and $Q = \Omega^{(1)} \times \cdots \times \Omega^{(r)} = \Pi_{k=1}^{r} \Omega^{(k)}$ is a hypercylindrical domain in the space E_n.

We concern ourselves next with the l-extension of a function $f(x)$ from a closed domain with sufficiently smooth boundaries. To do this we follow the procedure of Whitney and Hestenes ([16], pp. 666–681), who suggested a method of extending continuously differentiable functions from a closed domain with preservation of their smoothness. V. M. Babič [17] has concerned himself with the l-extension of functions from $W_p^{(l)}(\Omega)$ in the case when l is integral. Developing the method of V. M. Babič we solve the problem in the general case.

Lemma 7. *Let $\Omega = \sigma \times [0, b]$ $(x' = (x_1, \cdots, x_{n-1}) \in \sigma,\ 0 \le x_n \le b)$ be a cylindrical domain of the space E_n, $f(x) \in W_2^{(l)}(\Omega)$ and let $\beta_0, \beta_1, \cdots, \beta_{[l]}$ satisfy the system of equations:*

$$\sum_{m=1}^{[l]} \left(-\frac{1}{m+1}\right)^q \beta_m = 1 \quad (q = 0, 1, \cdots, [l]). \tag{40.1}$$

We denote by Ω' the domain of the space E_n, symmetrical to the domain Ω relative to the hyperplane $x_n = 0$. We put, in Ω',

$$\tilde{f}(x) = \sum_{m=0}^{[l]} \beta_m f\left(x', -\frac{x_n}{m+1}\right). \tag{41.1}$$

Then the function

$$f^*(x) = \begin{cases} f(x) & \text{for } x \in \Omega \\ \tilde{f}(x) & \text{for } x \in \Omega' \end{cases}$$

is an l-extension of the function $f(x)$ from Ω to $\Omega_1 = \Omega + \Omega'$. In addition

$$\|f^*\|_{W_2^{(l)}(\Omega_1)} \le C \|f\|_{W_2^{(l)}(\Omega)}, \tag{42.1}$$

where C is a positive constant which does not depend on $f(x)$.

Proof. Let p_1, p_2 be nonnegative integers whose sum p does not exceed l, and let $\psi(x)$ be a sufficiently smooth function which is defined in Ω_1 and vanishes on its boundary strip.

Integrating by parts p_1 times with respect to x_n we obtain

$$\begin{aligned}\int_\Omega \psi \frac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} f\,dx &= (-1)^{p_1} \int_\Omega \frac{\partial^{p_1}\psi}{\partial x_n^{p_1}} D_{x'}^{p_2} f\,dx \\ &- \sum_{q=0}^{p_1-1} (-1)^q \int_\sigma \frac{\partial^{p_1-q-1}\psi}{\partial x_n^{p_1-q-1}} \frac{\partial^q}{\partial x_n^q} D_{x'}^{p_2} f\,dx',\end{aligned} \tag{43.1}$$

$$\begin{aligned}\int_{\Omega'} \psi \frac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} \tilde{f}\,dx &= (-1)^{p_1} \int_{\Omega'} \frac{\partial^{p_1}\psi}{\partial x_n^{p_1}} D_{x'}^{p_2} \tilde{f}\,dx \\ &+ \sum_{q=0}^{p_1-1} (-1)^q \int_\sigma \frac{\partial^{p_1-q-1}\psi}{\partial x_n^{p_1-q-1}} \frac{\partial^q}{\partial x_n^q} D_{x'}^{p_2} \tilde{f}\,dx'.\end{aligned} \tag{44.1}$$

From (40.1) and (41.1) we have

$$\frac{\partial^q}{\partial x_n^q} D_{x'}^{p_2} \tilde{f} \Big|_{x_n=0} = \sum_{m=0}^{[l]} \beta_m \left(-\frac{1}{m+1}\right)^q \frac{\partial^q}{\partial x_n^q} D_{x'}^{p_2} f \Big|_{x_n=0}$$

$$= \frac{\partial^q}{\partial x_n^q} D_{x'}^{p_2} f \Big|_{x_n=0}. \tag{45.1}$$

By using the equations (43.1)–(45.1) and by integrating by parts with respect to x' we obtain:

$$\int_\Omega \psi \frac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} f\, dx + \int_{\Omega'} \psi \frac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} \tilde{f}\, dx = (-1)^{p_1} \int_\Omega \frac{\partial^{p_1}\psi}{\partial x_n^{p_1}} D_{x'}^{p_2} f\, dx$$

$$+ (-1)^{p_1} \int_{\Omega'} \frac{\partial^{p_1}\psi}{\partial x_n^{p_1}} D_{x'}^{p_2} \tilde{f}\, dx = (-1)^p \int_{\Omega_1} \frac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_1} \psi f^*\, dx.$$

It follows from this equation that the generalized derivative

$$\frac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} f^* = \begin{cases} \dfrac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} f & \text{when } x \in \Omega \\[2ex] \dfrac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} \tilde{f} & \text{when } x \in \Omega' \end{cases}$$

exists.

It is obvious that

$$\left\| \frac{\partial^{p_1}}{\partial x_n^{p_1}} D_x^{p_2} f^* \right\|_{L_2(\Omega_1)} \leq C \left\| \frac{\partial^{p_1}}{\partial x_n^{p_1}} D_{x'}^{p_2} f \right\|_{L_2(\Omega)} \tag{46.1}$$

Thus we have established the truth of the lemma when l is integral.

Def (Partition of unity)

Let $\Omega \subset \mathbb{R}^n$ be an open subset covered by a countable collection $\{\Omega_j\}_{j\in\mathbb{N}}$ of open subsets Ω_j.

A countable set of functions $\{\eta_j\}_{j\in\mathbb{N}}$ is a locally finite partition of unity (PoU) ~~subordinate~~ to the cover $\{\Omega_j\}$ if

(i) $\eta_j \in C_c^\infty(\Omega_j)$

(ii) $\eta_j \geq 0$, $\sum \eta_j = 1$ in Ω

(iii) at each point of Ω there is a neighbourhood on which only a finite number of η_j are non-zero.

Lemma 1. If $\Gamma \subset \mathbb{R}^n$ is a compact subset of $\mathbb{R}^n$ and $U_1, \dots, U_k$ be an open cover of Γ, i.e. $\Gamma \subset \bigcup_{i=1}^k U_i$, then there exists a PoU (η_j) subordinate to the cover $\{U_1, \dots, U_k\}$.

Lemma (Change of variables formula for Sobolev functions)

Let $\psi : \Omega \to \tilde{\Omega}$ be a bijective map between open $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ with $\psi \in C^{0,1}(\Omega)$ and $\psi^{-1} \in C^{0,1}(\tilde{\Omega})$.

If $u \in W^{1,p}(\Omega)$ and $v = u \circ \psi^{-1}$, then $v \in W^{1,p}(\tilde{\Omega})$ and

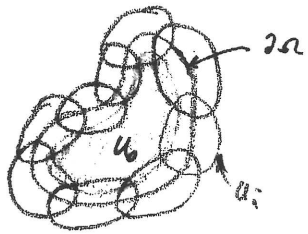
$$\frac{\partial u}{\partial x^i}(x) = \sum_{j=1}^n \frac{\partial v}{\partial y_j}(y) \frac{\partial \psi^j}{\partial x^i}(x) \quad \text{a.e. } x \in \Omega,$$

where $y = \psi(x) \in \tilde{\Omega}$ a.e.

PROOF OF EXTENSION THEOREM

1. <u>"Rectify" $\Gamma = \partial\Omega$ by local charts and use a PoU.</u>

As $\partial\Omega$ is compact and of class $C^{k-1,1}$, there exists open sets $\{U_1, \dots, U_k\}$ in Ω' s.t. $\partial\Omega \subset \bigcup_{i=1}^{k} U_i$ and bijective maps $\psi_i : U_i \to Q$ s.t.



(i) $\psi_i(U_i \cap \Omega) = Q_+$

(ii) $\psi_i(U_i \cap \partial\Omega) = Q_0$

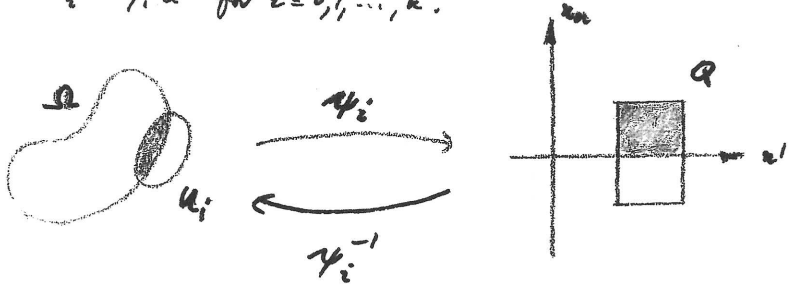
(iii) $\psi_i \in C^{k-1,1}(U_i)$, $\psi_i^{-1} \in C^{k-1,1}(Q)$.

Furthermore, we let $U_0 \subset\subset \Omega$ be a subdomain of Ω so that $\{U_0, U_1, \dots, U_k\}$ is a finite cover of Ω. Then let $\{\eta_i\}_{i=1}^{k}$ be a PoU subordinate to this cover.

2. Given $u \in W^{k,p}(\Omega)$ write

$$u = u_0 + u_1 + \dots + u_k$$

where $u_i = \eta_i u$ for $i = 0, 1, \dots, k$.



Then for $i = 1, \dots, k$ we note that

$$(\eta_i u) \circ \psi_i^{-1} \in W^{k,p}(Q_+) \quad \text{[by change of variables]}$$

$$\Rightarrow \quad E_0\left((\eta_i u) \circ \psi_i^{-1}\right) \in W^{k,p}(Q) \quad \text{[by Theorem A]}$$

$$\Rightarrow \quad E_0\left((\eta_i u) \circ \psi_i^{-1}\right) \circ \psi_i \in W_0^{k,p}(U_i) \quad \text{[change of variables, NB: supp } \eta_i \subset U_i]$$

Therefore the map E defined by

$$Eu = u\eta_0 + \sum_{i=1}^{k} E_0\left(\left((\eta_i u) \circ \psi_i^{-1}\right)\right) \circ \psi_i, \quad u \in W^{k,p}(\Omega),$$

satisfies $Eu \in W_0^{k,p}(\Omega')$, $Eu = u$ in Ω, and

$$\| Eu \|_{W^{k,p}(\Omega')} \le c \, \|u\|_{W^{k,p}(\Omega)},$$

where $c = c(k, N, \psi_i, \eta_i) = c(k, \Omega, \Omega') > 0$. ∎