§ Trace operator : Part I.

We want to assign to any $u \in W^{1,p}(\Omega)$ its "boundary values" along $\partial\Omega$. If $u \in C(\bar{\Omega})$, then u has boundary values in the usual sense. However $u \in W^{1,p}(\Omega)$ is only defined a.e. in Ω and as $\partial\Omega$ is a $\mathcal{L}^n$-measurable set of measure zero, there is no direct meaning one can give to "u restricted to $\partial\Omega$".

The notion of a "trace operator" resolves this problem.

Theorem (trace)

Let $\Omega \subset \mathbb{R}^n$ be a $C^{0,1}$-domain and $1 \le p \le \infty$.
Then there exists a bounded linear operator (i.e. the "trace operator")

$$T: W^{1,p}(\Omega) \to L^p(\partial\Omega, \mathcal{H}^{n-1})$$

s.t.

(1) $Tu = u|_{\partial\Omega}$ for $u \in C(\bar{\Omega}) \cap W^{1,p}(\Omega)$

(2) $\|Tu\|_{L^p(\partial\Omega)} \le C(n,p,\Omega)\|u\|_{W^{1,p}(\Omega)}$ for all $u \in W^{1,p}(\Omega)$.

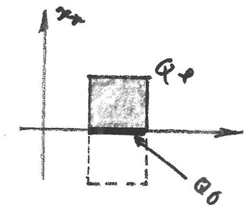
Proof. An exercise using PoU and "rectifying" the boundary together with the below result. ∎

Remark. There is a fundamental difference between $W^{1,p}(\Omega)$ and $L^p(\Omega)$ in that functions in $L^p(\Omega)$ do not have a trace on $\partial\Omega$,

Lemma. For $1 \le p < \infty$ there exists a bounded linear operator

$$T: W^{1,p}(Q_+) \to L^p(Q_0)$$

s.t. $Tu = u|_{Q_0}$ for all $u \in C^1(\bar{Q}_+) \cap W^{1,p}(Q_+)$.



Proof. Let $u \in C^1(\bar{Q})$. Then for $x' \in Q_0$ the FTC implies that

$$u(x', x_n) - u(x', 0) = \int_0^{x_n} \frac{\partial u}{\partial x_n}(x', s)\,ds.$$

Then by taking abs. values and integrating over $0 < x_n < 1$ we find that

$$|u(x',0)| \le \int_0^1 |u(x', x_n)|\,dx_n + \int_0^1 |Du(x', s)|\,ds.$$

By applying Jensen's inequality and integrating over $x' \in Q_0$ we get

$$\int_{Q_0} |u(x',0)|^p\,dx' \le \int_0^1 \left(\int_{Q_0} |u(x', x_n)|^p\,dx' \right) dx_n + \int_0^1 \left(\int_{Q_0} |Du(x', x_n)|^p\,dx' \right) dx_n$$

$$= \int_{Q_+} |u|^p\,dx + \int_{Q_+} |Du|^p\,dx. \quad \text{— (1)}$$

Now for $u \in W^{1,p}(Q)$ there exists $(u_k) \subset W^{1,p}(Q) \cap C^1(\bar{Q})$

s.t. $\quad \| u_k - u \|_{W^{1,p}(Q)} \to 0$ as $k \to \infty$.

From (1) we have

$$\| u_k|_{Q_0} - u_l|_{Q_0} \|_{L^p(Q_0)} \leq \| u_k - u_l \|_{W^{1,p}(Q_+)}$$

So that $\{u_k|_{Q_0}\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^p(Q_0)$.

We can then define $Tu = \lim_{k \to \infty} u_k|_{Q_0}$, in the L^p-sense, since $L^p(Q_0)$ is complete and T is bounded (i.e. by the BLT theorem).

Finally, for $u \in W^{1,p}(Q_+)$ there exists $\tilde{u} \in W^{1,p}(Q)$ by Theorem A and the result then follows by the above. ∎

Theorem (Characterisation of $W^{1,p}_0(\Omega)$ in terms of the trace)

Let $\Omega \subset \mathbb{R}^n$ be a $C^{0,1}$-domain, $1 \leq p < \infty$ and suppose $u \in W^{1,p}(\Omega)$.

Then $\quad u \in W^{1,p}_0(\Omega) \iff Tu = 0$ on $\partial\Omega$.

[i.e. the kernel of the trace operator

$\ker(T) = \{ u \in W^{1,p}(\Omega) : Tu = 0 \}$

is equal to $W^{1,p}_0(\Omega)$.]

Proof. 1. Suppose $u \in W^{1,p}_0(\Omega)$. Then by definition there exists $(u_k) \subset C^\infty_c(\Omega)$ s.t. $u_k \to u$ in $W^{1,p}(\Omega)$.

As $Tu_k = 0$ on $\partial\Omega$ for each $k \in \mathbb{N}$ and $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is a bounded linear operator, it follows that $Tu = 0$ on $\partial\Omega$.

(i.e. T is continuous w.r.t. strong convergence in $W^{1,p}(\Omega)$)

2. Now assume that $Tu = 0$ on $\partial\Omega$.

By a PoU and flattening out the boundary argument we can assume:

(i) $\boxed{u \in W^{1,p}(Q_+)}$

(ii) $Tu = 0$ on Q_0

(iii) $\mathrm{supp}(u) \subset Q_+$ is compact.

Then as $Tu = 0$ on Q_0 there exists $(u_k) \subset C^1(\bar{Q}_+)$ (by construction!) s.t.

$$u_k \to u \text{ in } W^{1,p}(Q_+) \quad \text{— (A)}$$

$$Tu_k = u_k|_{Q_0} \to 0 \text{ in } L^p(Q_0) \quad \text{— (B)}.$$

⌈ WTS: $\exists\, (v_\epsilon) \subset C_c^\infty(Q_+)$ s.t. $v_\epsilon \to u$ in $W^{1,p}(Q_+)$ from which we can conclude $u \in W_0^{1,p}(Q_+)$ ⌋

3. Let $\eta \in C^\infty(\mathbb{R}_+)$ be a bump function s.t.

$$\eta = \begin{cases} 1 & \text{on } [0,1] \\ 0 & \text{on } \mathbb{R}_+ \setminus [0,2] \end{cases}, \quad 0 \le \eta \le 1.$$

Then set $\eta_k(t) = \eta(kt)$ and put

$$w_k(x) := (1 - \eta_k(x_n))\, u(x).$$

Then $w_k \in W^{1,p}(Q_+)$ and $w_k = 0$ for $0 < x_n < \frac{1}{2k}$.

Note that

$$\begin{cases} \dfrac{\partial}{\partial x_n} w_k = \dfrac{\partial u}{\partial x_n}(1 - \eta_k) - k\eta' u \\[2mm] D_{x'} w_k = (1 - \eta_k) D_{x'} u \end{cases}$$

so we get

$$\int_{Q_+} |Dw_k - Du|^p dx \le \int_{Q_+} |D_{x'} w_k - D_{x'} u|^p dx + \int_{Q_+} |\partial_{x_n} w_k - \partial_{x_n} u|^p dx$$

$$\le c \underbrace{\int_{Q_+} |\eta_k|^p |Du|^p dx}_{\substack{\downarrow \text{(as } k \to \infty) \\ 0}} + \underbrace{C k^p \int_0^{2/k} ds \int_{Q_0} |u|^p dx'}_{= (*)}$$

(since $\eta_k \neq 0$ when $0 < x_n < \frac{2}{k}$)

Now if $x' \in Q_0$ and $0 < x_n < 1$ we have

$$|u_k(x', x_n)| \le |u_k(x', 0)| + \int_0^{x_n} \left| \frac{\partial u_k}{\partial x_n}(x', s) \right| ds$$

and so

$$\int_{Q_0} |u_k(x', x_n)|^p dx' \le c \left(\int_{Q_0} |u_k(x', 0)|^p dx' + x_n^{p-1} \int_0^{x_n} \int_{Q_0} |Du_k(x', s)|^p dx' ds \right).$$

Then taking $k \to \infty$ and using (A), (B) yields

$$\int_{Q_0} |u(x', x_n)|^p dx' \le c\, x_n^{p-1} \int_0^{x_n} \int_{Q_0} |Du|^p dx' ds \quad \text{— (**)}$$

for a.e. $x_n > 0$.

Now by using (**) to estimate (*) we find that

$$(*) \le c h^p \int_0^{2/h} dt \left(t^{p-1} \int_0^t \int_{Q_0} |Du|^p dx' ds \right)$$

$$\le c h^p \left(\int_0^{2/h} t^{p-1} dt \right) \left(\int_0^{2/h} ds \int_{Q_0} |Du|^p dx' \right)$$

$$\le c \int_0^{2/k} dx_n \int_{Q_0} |Du|^p dx'$$

$$\to 0 \text{ as } h \to \infty.$$

Therefore $\int_{Q_+} |Dw_h - Du|^p dx \to 0$ as $h \to \infty$.

and as $w_h \to u$ in $L^p(Q_+)$ also, we conclude $\boxed{w_h \to u \text{ in } W^{1,p}(Q_+).}$

4. Now by using a Poll and mollification (as done in the Sobolev density result of Lecture 16) we can find

$\tilde{w}_h \in C_c^\infty(Q_+)$ s.t. $\| \tilde{w}_h - w_h \|_{W^{1,p}(Q_+)} \le \frac{1}{h}$

(since $w_h = 0$ on $0 < x_n < \frac{1}{2k}$)

Hence we can conclude $u \in W_0^{1,p}(Q_+)$. ∎