§ Potential Estimates and Imbedding theorems

Let $0 \le \mu \le 1$. Define the operator V_μ on $L^1(\Omega)$ by the Riesz potential

$$V_\mu f(x) = \int_\Omega |x-y|^{n(\mu-1)} f(y)\,dy.$$

Lemma 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set and $x \in \mathbb{R}^n$. Then

$$\int_\Omega |x-y|^{n(\mu-1)}\,dy \le K|\Omega|^\mu,$$

where $K = K(n,\mu) = \frac{1}{\mu}\,\omega_n^{1-\mu}$. (†)

Proof. We may suppose that $x=0$. Then we find some $R>0$ such that $|B_R(0)| = |\Omega|$.

Then as

$$|\Omega \setminus B_R| = |\Omega| - |\Omega \cap B_R| = |B_R| - |\Omega \cap B_R| = |B_R \setminus \Omega|$$

we find that

$$\int_{\Omega\setminus B_R} |y|^{n(\mu-1)}\,dy \le R^{n(\mu-1)}|\Omega\setminus B_R| \quad (\text{as } |y|>R \text{ on } \Omega\setminus B_R)$$
$$= R^{n(\mu-1)}|B_R\setminus\Omega|$$
$$\le \int_{B_R\setminus\Omega} |y|^{n(\mu-1)}\,dy. \quad (\text{as } |y|<R \text{ on } B_R\setminus\Omega)$$

(†) NB: ω_n = volume of the unit ball in $\mathbb{R}^n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$.

in which case we find that

$$\int_\Omega |y|^{n(\mu-1)}\,dy \le \int_{B_R} |y|^{n(\mu-1)}\,dy$$
$$= |S^{n-1}| \int_0^R r^{n(\mu-1)} r^{n-1}\,dr$$
$$= \omega_n \frac{R^{n\mu}}{\mu}$$
$$\le \frac{\omega_n^{1-\mu}}{\mu} |\Omega|^\mu,$$

since $|\Omega| = |B_R| = n\omega_n R^n$. (NB: $|S^{n-1}| = n\,\omega_n$ is the area of the sphere) ∎

Lemma 2. The operator V_μ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ for any $1 \le q \le \infty$ satisfying

$$0 \le \delta = \delta(p,q) = \frac{1}{p} - \frac{1}{q} < \mu.$$

Furthermore, for any $f \in L^p(\Omega)$,

$$\|V_\mu f\|_{q;\Omega} \le C\,\|f\|_{p;\Omega},$$

where $C = \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta}$.

Proof. Choose $r \ge 1$ such that

$$\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta \qquad \text{— (*)}$$

Then $h(x-y) := |x-y|^{n(\mu-1)} \in L^q(\Omega)$, i.e.

$$\left(\int_\Omega |x-y|^{n(\mu-1)r}\, dy\right)^{1/r} = \left(\int_\Omega |x-y|^{n(\tilde{\mu}-1)}\, dy\right)^{1/r}$$

$$\le \tilde{K}^{1/r} |\Omega|^{\tilde{\mu}/r} \quad \text{(by Lemma 7)}$$

$$= \tilde{K}^{1/r} |\Omega|^{\mu-\delta},$$

where $\tilde{\mu} = (\mu-1)r+1$ and $\tilde{K} = \left(\frac{1-\delta}{\mu-\delta}\right) \omega_n^{-\left(\frac{\mu-1}{1-\delta}\right)}$.

Now re-write (*) as

$$\frac{1}{p'} + \frac{1}{q} + \frac{1}{r'} = 1 \qquad \text{— (**)}$$

where $\frac{1}{p'} + \frac{1}{p} = 1$ and $\frac{1}{r'} + \frac{1}{r} = 1$.

Moreover, as (**) implies that $\frac{p}{q} + \frac{p}{r'} = 1$, $\frac{r}{p'} + \frac{r}{q} = 1$

we can write

$$h|f| = h^{r/p'} h^{r/q} |f|^{p/q} |f|^{p/r'}$$

$$= (h^r |f|^p)^{1/q}\, h^{r/p'} |f|^{p/r'}.$$

Then by Hölder's inequality

$$|V_\mu f(x)| \le \int_\Omega |x-y|^{n(1-\mu)} |f(y)|\, dy$$

$$\le \left(\int h^r f^p\right)^{1/q} \left(\int h^r\right)^{1/p'} \left(\int f^p\right)^{1/r'}$$

$$= \left(\int_\Omega |x-y|^{n(\mu-1)r} |f(y)|^p dy\right)^{1/q} \left(\int_\Omega |x-y|^{n(\mu-1)r} dy\right)^{1/p'} \times \left(\int_\Omega |f(y)|^p dy\right)^{1/r'}$$

and as

$$\int_\Omega \int_\Omega |x-y|^{n(\mu-1)r} |f(y)|^p\, dy\, dx$$

$$= \int_\Omega dx\, |f(x)|^p \left(\int_\Omega dy\, |x-y|^{n(\mu-1)r}\right)$$

$$\le \tilde{K} |\Omega|^{(\mu-1)r+1} \left(\int_\Omega |f(x)|^p dx\right)$$

we find that

$$\|V_\mu f\|_q \le \left(\tilde{K} |\Omega|^{(\mu-1)r+1}\right)^{1/q} \left(\tilde{K} |\Omega|^{(\mu-1)r+1}\right)^{1/p'} \|f\|_p$$

$$\le \tilde{K}^{1/r} |\Omega|^{\mu-1+1/r} \|f\|_p$$

$$\le \tilde{K}^{1-\delta} |\Omega|^{\mu-\delta} \|f\|_p. \qquad \blacksquare$$

An equivalent formulation of Lemma 2 originally took the following form:

<u>Lemma</u> (Hardy – Littlewood – Sobolev)

For $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)\, v(y)}{|x-y|^{\lambda}}\, dx\, dy \leq c\, \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)}$$

for some $c = c(n,p,q) > 0$, where

$$\lambda = n\left(2 - \frac{1}{p} - \frac{1}{q}\right), \quad n \geq 1, \quad \frac{1}{p} + \frac{1}{q} < 1, \quad p > 1, \quad q > 1.$$

<u>Remark</u>. This was proved by Hardy & Littlewood (1928) when $n=1$ and by Sobolev (1938) for $n > 1$.

[Sob38] ———, *On a theorem of functional analysis*, Mat. Sb. (N.S.) **4 (46)** (1938), 471–497. (English transl. in Amer. Math. Soc. Transl. (2) **34** (1963), 39–68).

ON A THEOREM OF FUNCTIONAL ANALYSIS

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In this paper we give the proof of a theorem of functional analysis which is important in various questions of the theory of partial differential equations. In an earlier paper of the author it was established that the space $L_2^{(\nu)}$, i.e., the space of functions of n variables whose derivatives up to the order ν are square-integrable, is a subspace of the space $C^{(\nu-[\frac{n}{2}]-1)}$, i.e., the space of functions which have continuous derivatives up to the order $\nu-[\frac{n}{2}]-1$ inclusive [1, 2].

Convergence of a certain sequence in $L_2^{(\nu)}$ automatically induces convergence in $C^{(\nu-[\frac{n}{2}]-1)}$.

This theorem made it possible to prove certain new properties of semi-harmonic functions and also to introduce certain improvements into the theory of non-linear hyperbolic equations [3]. The results of the author were generalized by V. I. Kondrašov [4], who established that $L_p^{(\nu)}$ is a subspace of $C^{(\nu-[\frac{n}{p}]-1)}$. The present paper is a supplement and development of this result. We prove in it that the space $L_p^{(\nu)}$ lies in every $L^{(\nu-l)}_{\frac{1}{\frac{1}{p}-\frac{l}{n}}}$. In a subsequent paper we shall show how this result may be used in the theory of non-linear hyperbolic partial differential equations.

This first chapter contains a generalization of an important inequality due to F. Riesz. In the second chapter we apply this inequality to the proof of a theorem. A brief exposition of the results of the present paper is contained in a note by the author [5] in the Doklady Akademii Nauk.

CHAPTER I

Generalization of some inequalities of F. Riesz

§1. Generalization of Riesz's lemma. Let there be given in n-dimensional space three functions

$$\phi_1(x_1, \cdots, x_n),\ \phi_2(x_1, \cdots, x_n),\ \phi_3(x_1, \cdots, x_n), \tag{1.1}$$

taking only the two values 0 and 1, i.e., characteristic functions of three sets E_1, E_2, and E_3. Let

$$\text{mes } E_1 < \infty,\ \text{mes } E_2 < \infty, \text{ and mes } E_3 < \infty. \tag{1.2}$$

§4. Generalization of the second integral inequality of Riesz. Let the functions $\phi(x_1, \cdots, x_n)$ and $\psi(y_1, \cdots, y_n)$ be pth power and qth power summable, respectively, over the entire space, where $p > 1$ and $q > 1$, $\frac{1}{p}+\frac{1}{q} > 1$. Then the integral

$$J = \overbrace{\int\cdots\int}^{2n} \frac{\phi(x_1, \cdots, x_n),\ \psi(y_1, \cdots, y_n)}{r^\lambda}\, dx_1 \cdots dx_n dy_1 \cdots dy_n, \tag{4.1}$$

where

$$r = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \text{ and } \lambda = n\left[2 - \frac{1}{p} - \frac{1}{q}\right] = n\left[\frac{1}{p'} + \frac{1}{q'}\right], \tag{4.2}$$

$$\frac{1}{p}+\frac{1}{p'} = 1,\quad \frac{1}{q}+\frac{1}{q'} = 1,$$

has meaning and satisfies the inequality

$$\overbrace{\int\cdots\int}^{2n} \frac{\phi(x_1, \cdots, x_n)\psi(y_1, \cdots, y_n)}{r^\lambda}\, dx_1 \cdots dx_n dy_1 \cdots dy_n$$
$$\leq K\left[\overbrace{\int\cdots\int}^{n} |\phi|^p dx_1 \cdots dx_n\right]^{\frac{1}{p}} \left[\overbrace{\int\cdots\int}^{n} |\psi|^q dy_1 \cdots dy_n\right]^{\frac{1}{q}}, \tag{4.3}$$

where the constant K depends on p and q, but does not depend on the functions ϕ and ψ.

For the proof we may assume from the beginning that $\phi > 0$ and $\psi > 0$. Moreover, by the previous lemma, we may suppose from the beginning that ϕ depends only on $r_1 = \sqrt{\sum_{i=1}^{n} x_i^2}$ and that ψ depends only on $r_2 = \sqrt{\sum_{i=1}^{n} y_1^2}$.

We break up the integral J into three terms. Let

CHAPTER II

Relation between the spaces $L_p^{(\nu)}$

§6. **A new definition of derivative.** Let $\phi(x_1, \cdots, x_n)$ be some function of the variables $x_1, \cdots, x_n$ which is summable in the domain D. Let D' be any other domain lying along with its boundary inside D.

Let us consider any function $\psi(x_1, \cdots, x_n)$ having continuous derivatives

up to the order α inclusive and vanishing outside of D'. If the function ϕ itself has continuous derivatives up to the order α, then the equation

$$\overset{n}{\int \cdots \int_D} \left\{ \phi \frac{\partial^\alpha \psi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} + (-1)^{\alpha-1} \psi \frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right\} dx_1 \cdots dx_n = 0 \tag{6.1}$$

is valid.

We shall consider this equation as the definition of the derivative and call derivative of ϕ,

$$\frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

such a function of the variables $x_1, \cdots, x_n$ which is summable over any bounded subset of the domain D and satisfies condition (6.1) for all ψ.

It is obvious that the derivative thus defined may only be unique, for supposing the existence of two such derivatives

$$\left[\frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right]_1 \quad \text{and} \quad \left[\frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right]_2,$$

we will have for any ψ

$$\overset{n}{\int \cdots \int_D} \psi \left\{ \left[\frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right]_1 - \left[\frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right]_2 \right\} dx_1 \cdots dx_n = 0, \tag{6.2}$$

from which it follows that these two derivatives are equal to each other everywhere except, perhaps, a set of measure zero.

Comparatively simple examples show that the derivative in our sense may exist when, for instance, the function does not have a derivative almost everywhere, and conversely the existence of the derivative almost everywhere does not necessitate the existence of the derivative in our sense.

In fact, let

$$\phi = f_1(x_1) + f_2(x_2), \tag{6.3}$$

where neither $f_1(x_1)$ nor $f_2(x_2)$ is differentiable. Then the derivative $\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$ does not exist from the classical point of view. Nevertheless, by our definition

$$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0 \tag{6.4}$$

at all points. Indeed,

$$\overbrace{\int\cdots\int}^{n}_{D} \phi \frac{\partial^2\psi}{\partial x_1 \partial x_2}\, dx_1 \cdots dx_n = \overbrace{\int\cdots\int}^{n}_{D} f_1(x_1) \frac{\partial}{\partial x_2}\left(\frac{\partial\psi}{\partial x_1}\right) dx_1 \cdots dx_n$$

$$+ \overbrace{\int\cdots\int}^{n}_{D} f_2(x_2) \frac{\partial}{\partial x_1}\left(\frac{\partial\psi}{\partial x_2}\right) dx_1 \cdots dx_n = 0, \qquad (6.5)$$

since both terms may be integrated by parts. On the other hand, let

$$\phi = F(x_1), \qquad (6.6)$$

where F has a summable derivative almost everywhere, but is not absolutely continuous. Then, according to our definition, the derivative does not exist, of which the reader may easily convince himself.

For discovering the existence of the derivative it is useful to point out one indication based on the theory of approximations.

Theorem. *If one can approximate a given summable function ϕ by a sequence of continuously differentiable functions*

$$\phi_k(x_1, \cdots, x_n) \quad (k = 0, 1, \cdots) \qquad (6.7)$$

such that for any function ψ, continuous and vanishing outside the domain D' which is interior to the domain D, the equation

$$\lim_{k\to\infty} \overbrace{\int\cdots\int}^{n}_{D} \phi_k\psi\, dx_1 \cdots dx_n = \overbrace{\int\cdots\int}^{n} \phi\psi\, dx_1 \cdots dx_n \qquad (6.8)$$

holds, and if, in addition, for any bounded subset D' of the domain D

$$\overbrace{\int\cdots\int}^{n}_{D'} \left| \frac{\partial^\alpha \phi_k}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p dx_1 \cdots dx_n \le A, \qquad (6.9)$$

where $p > 1$, then the derivative $\frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ exists and satisfies the condition

$$\overbrace{\int\cdots\int}^{n}_{D'} \left| \frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p dx_1 \cdots dx_n \le A. \qquad (6.10)$$

For the proof we shall use the theorem on weak compactness of the unit sphere of the space L_p of functions which are pth power summable (see [10], p. 126).

By this weak compactness there exists a function $\omega_{\alpha_1, \cdots, \alpha_n}$ and a subsequence

$$\frac{\partial^\alpha \phi_{k_i}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad i = 1, 2, \cdots, \qquad (6.11)$$

such that for any function ψ which is p'th power summable on D', $p' = \frac{p}{p-1}$, the equation

$$\lim_{i\to\infty} \overbrace{\int\cdots\int}^{n}_{D'} \psi \frac{\partial^\alpha \phi_{k_i}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\, dx_1 \cdots dx_n = \overbrace{\int\cdots\int}^{n}_{D'} \psi\omega_{\alpha_1, \cdots, \alpha_n}\, dx_1 \cdots dx_n \qquad (6.12)$$

is valid. Moreover,

$$\overbrace{\int\cdots\int}^{n}_{D'} |\omega_{\alpha_1, \cdots, \alpha_n}|^p\, dx_1 \cdots dx_n \le A. \qquad (6.13)$$

Let us write the identity

$$\overbrace{\int\cdots\int}^{n}_{D'} \psi \frac{\partial^\alpha \phi_{k_i}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\, dx_1 \cdots dx_n$$

$$= (-1)^\alpha \overbrace{\int\cdots\int}^{n}_{D'} \phi_{k_i} \frac{\partial^\alpha \psi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\, dx_1 \cdots dx_n, \qquad (6.14)$$

where ψ is a continuously differentiable function equal to zero outside D'. Passing to the limit in equation (6.14) we obtain

$$\overbrace{\int\cdots\int}^{n}_{D'} \psi\omega_{\alpha_1, \cdots, \alpha_n}\, dx_1 \cdots dx_n = (-1)^\alpha \overbrace{\int\cdots\int}^{n}_{D'} \phi \frac{\partial^\alpha \psi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\, dx_1 \cdots dx_n. \qquad (6.15)$$

Hence it follows that the function $\omega_{\alpha_1, \cdots, \alpha_n}$ is the derivative of ϕ, as was to be proved.

§7. **Fundamental theorem for the case of continuous functions.** Let us consider some domain D in the space $x_1, \cdots, x_n$. Let this domain be such that any of its interior or boundary points can be reached by the point of a moving spherical sector of constant magnitude and shape. (D may be unbounded.)

Let us consider in this domain some function $\phi(x_1, \cdots, x_n)$ with respect to which we shall make the following assumptions:

(1) *the function $|\phi(x_1, \cdots, x_n)|^{q^*}$ is summable on D, where $q^* \ge 1$,*

(2) *the function $\phi(x_1, \cdots, x_n)$ admits all derivatives of order l in the domain D,*

(3) *all derivatives of order l are pth power summable*, $p \geq 1$.

$$\overbrace{\int \cdots \int}^{n}_{D} \left| \frac{\partial^l \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p dx_1 \cdots dx_n \leq A^p, \tag{7.1}$$

where $l \leq n$.

In addition, several possibilities are meaningful:

$$1)\ 1 < p < \frac{n}{l}, \qquad 3)\ 1 < p = \frac{n}{l},$$
$$2)\ 1 = p < \frac{n}{l}, \qquad 4)\ 1 = p = \frac{n}{l}, \tag{7.2}$$
$$5)\ p > \frac{n}{l}.$$

The last case has been the object of special investigation. For $p = 2$ it was analyzed by the author [1, 2], and for the case of arbitrary p by V. I. Kondrašov, who used the same method. There it is proved that the function $\phi(x_1, \cdots, x_n)$ will be continuous and will satisfy Hölder's condition with certain positive exponents. Interesting results concerning this question were obtained by I. G. Petrovskiĭ and K. N. Smirnov [11].

Our task will be the consideration of the remaining cases. First of all, we shall study the first of these. Let us restrict ourselves only to those cases when $\frac{1}{q^*} > \frac{1}{p} - \frac{1}{n}$. Then we have the

Theorem. *If* $1 < p < \frac{n}{l}$, *then the function* ϕ *is qth power summable on the domain D, where*

$$\frac{1}{q} = \frac{1}{p} - \frac{l}{n}, \tag{7.3}$$

and, moreover,

$$\left[\overbrace{\int \cdots \int}^{n}_{D} |\phi|^q dx_1 \cdots dx_n \right]^{\frac{1}{q}} \leq NA^p + L \left[\overbrace{\int \cdots \int}^{n}_{D} |\phi|^{q^*} dx_1 \cdots dx_n \right]^{\frac{1}{q^*}}, \tag{7.4}$$

where the constants N and L depend on the shape of the domain D and on the exponents p, q^*, k, n and 1, but do not depend on the form of the function ϕ.*

*If instead of the fundamental theorem of the previous chapter we used its weakened form (see previous footnote), then we would obtain a weaker result in this case. We would prove the summability of the function ϕ only for the power $q^{**} < q$. This weak result is sometimes sufficient for the applications.

BIBLIOGRAPHY

[1] S. L. Sobolev, *On some estimates of families of functions having square-integrable derivatives*, Dokl. Akad. Nauk SSSR 1 (1936), 267–270. (Russian)

[2] ———, *Sur un problème limite pour les équations polyharmoniques*, Mat. Sb. (N. S.) 2 (44) (1937), 467–500. (Russian. French summary)

[3] F. Frankl, *Über das Anfangswertproblem für lineare und nichtlineare hyperbolische Differentialgleichungen zweiter Ordnung*, Mat. Sb. (N. S.) 2 (44) (1937), 793–814. (Russian. German summary)

[4] V. I. Kondrašov, *On some estimates of families of functions subject to an integral inequality*, Dokl. Akad. Nauk SSSR 18 (1938), 235–240. (Russian)

[5] S. L. Sobolev, *On a theorem of functional analysis*, Dokl. Akad. Nauk SSSR 20 (1938), 5–10. (Russian)

[6] F. Riesz, *Sur une inégalité intégrale (Extrait d'une lettre adressée à Hardy)*, J. London Math. Soc. 5 (1930), no. 19, 162–168.

[7] L. A. Ljusternik, *The Brunn-Minkowski inequality for arbitrary measurable sets*, Dokl. Akad. Nauk SSSR 3 (1935), 55–58. (Russian)

[8] W. Blaschke, *Kreis und Kugel*, Viet, Leipzig, 1916; Reprint, Chelsea, New York, 1949.

[9] G. Hardy, J. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, London, 1934.

[10] L. A. Ljusternik, *Fundamental concepts of functional analysis*, Uspehi Mat. Nauk 1 (1936), 77–140. (Russian)

[11] I. G. Petrovskiĭ and K. N. Smirnov, *On conditions of equicontinuity of families of functions*, Bjull. Moskov. Univ. Ser. A 1 (1938), no. 8. (Russian)

[12] *General theory of diffraction of waves on Riemann surfaces*, Trudy Fiz.-Mat. Inst. Steklov. 9 (1935), 39–108. (Russian)