We can now use the fact that V_μ is a bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$ for any $1 \le q \le \infty$ s.t. $0 \le \frac{1}{p} - \frac{1}{q} < \mu$.

Lemma 3. Let $u \in W_0^{1,1}(\Omega)$. Then

$$u(x) = \frac{1}{n\omega_n} \int_\Omega \frac{(x_i - y_i) D_i u(y)}{|x-y|^n}\, dy \quad \text{a.e. } (\Omega).$$

Proof. Suppose $u \in C_c^\infty(\Omega)$ and extend $u = 0$ on $\mathbb{R}^n \setminus \Omega$.

Then for any $\omega \in S^{n-1}$,

$$u(x) = -\int_0^\infty \frac{d}{dt} u(x + t\omega)\, dt.$$

Hence

$$u(x) = \frac{-1}{|S^{n-1}|} \int_0^\infty \int_{\partial B(0,1)} \frac{d}{dt} u(x+t\omega)\, d\sigma(\omega)\, dt$$

$$= \frac{-1}{|S^{n-1}|} \int_0^\infty \int_{\partial B(0,1)} D_i u(x+t\omega)\, \omega^i\, d\sigma(\omega)\, dt$$

$$= \frac{-1}{|S^{n-1}|} \int_0^\infty \int_{\partial B(y,t)} Du(y) \frac{y_i - x_i}{|y-x|^n}\, d\sigma(y)\, dt, \quad y = x + t\omega,\ |y-x| = t.$$

$$= \frac{1}{n\omega_n} \int_\Omega Du(y) \frac{x_i - y_i}{|x-y|^n}\, dy \quad \text{(by the co-area formula)}.$$

The desired result now follows from the density theorem. ∎

Corollary For $u \in W^{1,1}(\Omega)$ we have

$$|u| \le \frac{1}{n\omega_n} V_{\frac{1}{n}} |Du| \quad \text{a.e. } (\Omega).$$

Lemma (Poincaré inequality, Version I)

For $u \in W_0^{1,p}(\Omega)$, $1 \le p < \infty$, we have

$$\|u\|_{p;\Omega} \le \left(\frac{|\Omega|}{\omega_n}\right)^{1/n} \|Du\|_{p;\Omega}.$$

Proof. From the Corollary we get

$$\|u\|_p \le \frac{1}{n\omega_n} \| V_{\frac{1}{n}} |Du| \|_p$$

$$\le \left(\frac{|\Omega|}{\omega_n}\right)^{1/n} \|Du\|_p \qquad \left(\mu = \frac{1}{n},\ \delta = 0\right)$$

where the last inequality follows by Lemma 2. ∎

Lemma 4. Let $\Omega \subset \mathbb{R}^n$ be convex and $u \in W^{1,1}(\Omega)$. Then

$$|u(x) - (u)_\Omega| \le \frac{d^n}{n} \fint_\Omega |x-y|^{1-n} |Du(y)|\,dy \quad \text{a.e. } (\Omega),$$

where $(u)_\Omega = \fint_\Omega u\,dx$ and $d = \operatorname{dist}(\Omega)$.

Proof. By the density theorem we can assume w.l.o.g. that $u \in C^1(\Omega)$. Then for $x, y \in \Omega$ we have

$$u(x) - u(y) = -\int_0^{|x-y|} \frac{d}{dt} u(x+t\omega)\,dt, \qquad \omega = \frac{x-y}{|x-y|}.$$

Fix $x \in \Omega$ and integrate w.r.t. $y \in \Omega$ to get

$$u(x) - (u)_\Omega = \frac{-1}{|\Omega|} \int_\Omega dy \int_0^\infty \frac{d}{dt} u(x+t\omega)\,dt. \qquad \text{—} (*)$$

Now write

$$U(z) = \begin{cases} \left|\frac{d}{dt} u(z)\right|, & z \in \Omega \\ 0, & z \notin \Omega \end{cases}$$

and split $y \in \Omega$ (as Ω = convex) into spatial coordinates

$$y = x + \rho\omega, \qquad \rho > 0 \ \& \ \omega \in S^{n-1},$$

and let $\delta(x,\omega) = \sup\{\rho : x + \rho\omega \in \Omega\}$.

i.e. we have $y = x + \rho\omega$, $\omega \in S^{n-1}$, $0 < \rho < \delta(x,\omega)$)

Then from $(*)$ we have

$$|u(x) - (u)_\Omega| \le \frac{1}{|\Omega|} \int_\Omega dy \int_0^\infty U(x+t\omega)\,dt, \qquad \omega = \frac{x-y}{|x-y|}$$

$$= \frac{1}{|\Omega|} \int_{S^{n-1}} d\omega \left(\int_0^{\delta(x,\omega)} \left(\int_0^\infty U(x+t\omega)\,dt \right) \rho^{n-1}\,d\rho \right), \qquad y = x + \rho\omega$$

$$= \frac{1}{|\Omega|} \int_0^\infty dt \left(\int_{S^{n-1}} d\omega \left(\int_0^{\delta(x,\omega)} U(x+t\omega)\rho^{n-1}\,d\rho \right)\right)$$

$$= \frac{1}{|\Omega|} \int_0^\infty dt \int_{S^{n-1}} d\omega \ U(x+t\omega) \frac{\delta(x,\omega)^n}{n}$$

$$\le \frac{d^n}{n} \frac{1}{|\Omega|} \int_{S^{n-1}} d\omega \int_0^\infty U(x+t\omega)\,dt$$

$$= \int_0^\infty dt \int_{\partial B(x,t)} U(z) \frac{d\sigma(z)}{t^{n-1}}, \qquad z = x + t\omega$$

$$= \int_\Omega \frac{|\partial_r u(z)|}{|z-x|^{n-1}} \mathbb{1}_\Omega(z)\,dz. \qquad \blacksquare$$

Lemma (Poincaré inequality, Version II)

For $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$ and convex Ω, we have

$$\|u - (u)_\Omega\|_{p;\Omega} \le \left(\frac{\omega_n d^n}{|\Omega|} \right)^{1-1/n} d\, \|Du\|_{p;\Omega}$$

where $d = \operatorname{diam}(\Omega)$.

Theorem (Morrey)

Let $u \in W_0^{1,p}(\Omega)$, $p > n$. Then $u \in C^{0,\alpha}(\bar{\Omega})$ and furthermore

$$|u(x) - u(y)| \le c(n,p) \|Du\|_{p;\Omega} |x-y|^\alpha, \qquad x, y \in \Omega,$$

where $\alpha = 1 - n/p$.

Proof By Lemma 2 and 4 $(q = \infty,\ \mu = \frac{1}{n})$

on a ball $B = B_R(x_0)$, $x_0 = \frac{x+y}{2}$, $R = \frac{1}{2}|x-y|$,

we have

$$|u(x) - (u)_B| \le c R^{n(\mu - \delta)} \|Du\|_p$$

$$\le c R^{1-n/p} \|Du\|_p \qquad \text{a.e. } (\Omega \cap B).$$

Then

$$|u(x) - u(y)| \le |u(x) - (u)_B| + |u(y) - (u)_B|$$

$$\le 2 c R^{1-n/p} \|Du\|_{p;\Omega} \qquad \text{a.e. } (\Omega \cap B). \quad \blacksquare$$

Corollary For $k \in \mathbb{N}$, $m \in \mathbb{N}_0$, $p \ge 1$ with $m < \frac{n}{p} < m+1$

we have

$$W_0^{k,p}(\Omega) \hookrightarrow C^{k-m-1,\, 1+m-n/p}(\bar{\Omega})$$

Proof. For $mp < n$, $W_0^{k,p}(\Omega) \hookrightarrow W_0^{k-m, \frac{np}{n-mp}}(\Omega)$ and

$D^{k-m-1} u \in C^{0, 1-\gamma}(\bar{\Omega})$, $\gamma = \frac{n}{p} - m$, since $(m+1)p > n$.

Thus $D^{k-m-1} u \in C^{0, 1+m-\frac{n}{p}}(\bar{\Omega})$. $\blacksquare$