WEAK SOLUTIONS & DIFFERENTIABILITY

§ Difference quotients

The differentiability of PDE solutions may be deduced through considering their difference quotients.

Let $u: \Omega (\subset \mathbb{R}^n) \to \mathbb{R}$ and let e_i = unit coordinate vector in x_i-direction.

Define

$$\Delta^h u(x) = \Delta_i^h u(x) = \frac{u(x+he_i) - u(x)}{h}, \quad h \neq 0.$$

Lemma 1. Let $u \in W^{1,p}(\Omega)$. Then $\Delta^h u \in L^p(\Omega')$ for all $\Omega' \subset\subset \Omega$ satisfying $h < \mathrm{dist}(\Omega', \partial\Omega)$ and

$$\|\Delta^h u\|_{L^p(\Omega')} \le \|D_i u\|_{L^p(\Omega)}.$$

Proof. Suppose initially that $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Then

$$\Delta^h u(x) = \frac{1}{h} \int_0^h D_i u(x_1, \dots, x_{i-1}, x_i + \xi, x_{i+1}, \dots, x_n)\, d\xi$$

so that by Hölder's inequality

$$|\Delta^h u(x)|^p \le \left(\frac{1}{h} \int_0^h |D_i u(\text{---}\,\text{''}\,\text{---})|\, d\xi \right)^p$$
$$\le \frac{1}{h} \int_0^h |D_i u(\text{---}\,\text{''}\,\text{---})|^p d\xi$$

and hence

$$\int_{\Omega'} |\Delta^h u|^p dx \le \frac{1}{h} \int_0^h \int_{B_h(\Omega')} |D_i u|^p \, dx\, d\xi \le \int_\Omega |D_i u|^p dx$$

$$B_h(\Omega') = \bigcup_{x \in \Omega'} \{ y : |y - x| < h \}$$

Then an approx. argument gives the result for $W^{1,p}$-functions.

Lemma 2. Let $u \in L^p(\Omega)$, $1 < p < \infty$, and suppose there exists $K > 0$ such that

$$\|\Delta^h u\|_{L^p(\Omega')} \le K$$

for all $h > 0$ and for all $\Omega' \subset\subset \Omega$ satisfying $h < \mathrm{dist}(\Omega', \partial\Omega)$.

Then the weak derivative $D_i u$ exists and satisfies $\|D_i u\|_{L^p(\Omega)} \le K$.

Proof. By weak compactness of the unit ball in $L^p(\Omega')$, $1 < p < \infty$, there exists $h_j \to 0$ and $v \in L^p(\Omega)$ with $\|v\|_p \le K$ s.t.

$$\int_\Omega \varphi \Delta^{h_j} u \, dx \to \int_\Omega \varphi v \, dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Now for $h_j < \mathrm{dist}(\mathrm{supp}\, \varphi, \partial\Omega)$ we have

$$\int_\Omega \varphi \Delta^{h_j} u \, dx = - \int_\Omega u \Delta^{-h_j} \varphi \, dx \to - \int_\Omega u D_i \varphi \, dx.$$

Hence,

$$\int_\Omega \varphi v \, dx = - \int_\Omega u D_i \varphi \, dx$$

from which we get $v = D_i u$. ∎

§ Weak solutions

Consider an operator L of the form

$$Lu = D_i(a^{ij} D_j u),$$

where a^{ij} are measurable functions on a domain $\Omega \subset \mathbb{R}^n$.

We say u satisfies $Lu = 0$ in a <u>weak sense</u> if

(†) $$B(u,\varphi) = \int_\Omega a^{ij} D_j u \, D_i \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

NB: φ is called a <u>test function</u>.

We assume that

(i) L is <u>strictly elliptic</u> in Ω, i.e. there exists $\lambda > 0$ s.t.

$$a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \ \& \text{ a.e. } x \in \Omega.$$

NB: λ is called the ellipticity constant.

(ii) L has bounded coefficients, i.e. for some $\Lambda > 0$,

$$a^{ij}(x) \leq \Lambda \quad \text{for a.e. } x \in \Omega.$$

Condition (ii) implies that

$$|B(u,\varphi)| \leq \int_\Omega |a^{ij} D_j u \, D_i \varphi| \, dx \leq c \, \|u\|_{W^{1,2}} \|\varphi\|_{W^{1,2}}.$$

Thus the map $\varphi \mapsto B(u,\varphi)$, for fixed $u \in W^{1,2}$, is a bounded linear functional on $W_0^{1,2}(\Omega)$.

So if (†) holds for $\varphi \in C_c^\infty$, then (†) holds for $\varphi \in W_0^{1,2}$.

<u>Theorem</u>. Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, where L is strictly elliptic, a^{ij} are uniformly Lipschitz continuous and $f \in L^2(\Omega)$. Then for any subdomain $\Omega' \subset\subset \Omega$, we have $u \in W^{2,2}(\Omega')$ and

$$\|D^2 u\|_{L^2(\Omega')} \leq c\left(\|Du\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

for some $c = c(n, \lambda, K, d')$, where $d' = \mathrm{dist}(\Omega', \partial\Omega)$ and $K = \max_{i,j} \|a^{ij}\|_{C^{0,1}(\bar\Omega)}$.

Proof of Theorem.

Let $\eta \in C_c^\infty(\Omega)$ satisfy $0 \le \eta \le 1$ and take

$$\varphi = \Delta^{-h}(\eta^2 \Delta^h u) \in W_0^{1,2}(\Omega) \text{ for } |h| < \tfrac{1}{2}\operatorname{dist}(\operatorname{supp}\eta, \partial\Omega).$$

Then by the weak formulation, we find that

$$\int_\Omega f\varphi\,dx = -\int_\Omega a^{ij} D_j u\, D_i\varphi\,dx$$

$$= \int_\Omega \underbrace{\Delta^h(a^{ij} D_j u)}\,\underbrace{D_i(\eta^2 \Delta^h u)}\,dx$$

$$= 2\eta D_i\eta\, \Delta^h u + \eta^2 D_i \Delta^h u$$

$$= a^{ij}(x+he_k)\Delta^h D_j u(x) + \Delta^h a^{ij}(x)\, D_j u(x)$$

(since $\Delta^h(fg)(x) = f(x+he_k)\Delta^h g(x) + \Delta^h f(x) g(x)$)

Now,

$$\int_\Omega f\varphi\,dx = \int_\Omega f\,\Delta^{-h}(\eta^2\Delta^h u)\,dx$$

$$\le \|f\|_2 \,\|\Delta^{-h}(\eta^2\Delta^h u)\|_2$$

$$\le \|f\|_2\, \|D(\eta^2 \Delta^h u)\|_2 \qquad \text{(by Lemma 1)}$$

$$\le c\|f\|_2 \left(\|D\eta \Delta^h u\|_2 + \|\eta \Delta^h Du\|_2\right).$$

We also estimate

$$\int_\Omega a^{ij}(x+he_k)\,\eta^2 \Delta^h D_j u\, \Delta^h D_i u\,dx \ge \lambda \int_\Omega |\eta \Delta^h Du|^2\,dx$$

(by the ellipticity condition)

as well as

$$\int_\Omega \Delta^h a^{ij}(x) D_j u(x)\, \eta^2 D_i \Delta^h u\,dx \le \int_\Omega K\,|Du|\,|\eta^2 D_i \Delta^h u|\,dx$$

$$\le K\|Du\|_2\, \|\eta \Delta^h Du\|_2$$

and

$$\int_\Omega \Delta^h(a^{ij} D_j u)(2\eta D_i\eta\, \Delta^h u)\,dx$$

$$\le \int_\Omega K(|\eta \Delta^h Du| + |Du|)\,|D\eta \Delta^h u|\,dx$$

$$\le C\|\eta \Delta^h Du\|_2 \|D\eta \Delta^h u\|_2 + C\|Du\|_2\|D\eta\Delta^h u\|_2 .$$

Therefore we can estimate

$$\lambda\|\eta \Delta^h Du\|_2^2 \le C\|\eta \Delta^h Du\|_2\left(\|Du\|_2 + \|D\eta \Delta^h u\|_2 + \|f\|_2\right) + C\|D\eta \Delta^h u\|_2\left(\|Du\|_2 + \|f\|_2\right)$$

Then by using Young's inequality (i.e. $ab \le \varepsilon a^2 + c(\varepsilon) b^2$) we get

$$\|\eta \Delta^h Du\|_2 \le c\left(\|Du\|_2 + \|f\|_2 + \|D\eta \Delta^h u\|_2\right)$$

Moreover, by lemma 1 we have

$$\|D\eta\, \Delta^h u\|_2 \leq \sup|D\eta| \cdot \|Du\|_2$$

so that we conclude

$$\boxed{\|\eta\, \Delta^h Du\|_2 \leq c\left[\left(1+\sup|D\eta|\right)\|Du\|_2 + \|f\|_2 \right].}$$

Now by choosing a cut-off function η such that $\eta = 1$ on $\Omega' \subset\subset \Omega$ and $|D\eta| \leq \frac{2}{d'}$, $d' = \operatorname{dist}(\Omega', \partial\Omega)$, we find by lemma 2 that $Du \in W^{1,2}(\Omega')$, $\forall\, \Omega' \subset\subset \Omega$, and the desired estimate holds. ∎