Calculus of Variations: Introduction

The model case takes the following form

$$\inf \left\{ I(u) = \int_a^b f(x, u, u_x)\, dx \ : \ u \in X \right\},$$

i.e. want to find a minimiser $u \in X$ s.t.

$$I(u) \le I(v), \quad v \in X,$$

where X = admissible class of functions.

e.g. Fermat Principle (1662)

"Find the trajectory of a light ray in an inhomogeneous medium so that the total time taken to travel between two given points is minimal."

Suppose we are given a medium such that the velocity of light $v = v(x,y) \in (0,1]$.

Between points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, $\quad x_1 < x_2$, $y_1 < y_2$,

we want to find the trajectory

$$y = y(x), \quad x \in [x_1, x_2],$$

that minimises the total time

$$T = \int_{t_1}^{t_2} dt = \int_A^B \frac{1}{v}\, ds = \int_{x_1}^{x_2} \frac{1}{v(x,y)} \sqrt{1 + y'^2}\, dx$$

NB: The index of refraction $n = \frac{1}{v}$.

i.e. we want to find

$$\inf \left\{ T(y) \ : \ y \in W^{1,1}(x_1, x_2) \text{ s.t. } y(x_1) = y_1 \ \& \ y(x_2) = y_2 \right\},$$

Jacob Bernoulli condition:

"any curve which minimises a given integral must have sub-arcs minimising the same integral".

The necessary condition has to be a local condition (i.e. involving differentials).

For the Fermat problem it is of the form:

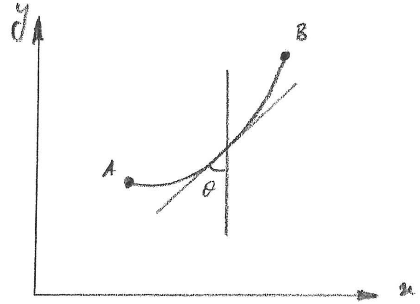
$$n(x,y)\, y'' + \left(n_x(x,y)\, y' - n_y(x,y) \right)(1 + y'^2) = 0 \qquad (*)$$

Remark. If $n = n(y)$, then $(*)$ can be re-written as

(Snell's law)

$$\frac{d}{dx}\left(\frac{\sqrt{1+y'^2}}{n(y)} \right) = 0 \quad \Rightarrow \quad \boxed{n(y) \sin\theta = \text{const.}}$$

where $\sin\theta = \frac{1}{\sqrt{1+y'^2}}$.



§ The Euler – Lagrange equations

Suppose a function $u \in X = \{u \in C^1[a,b] : u(a)=\alpha,\ u(b)=\beta\}$ minimises the integral $I(u)$ among all such functions.

Then for any $\varphi \in C^1_c[a,b]$ the function $u + t\varphi \in X$.

If we define

$$\Phi(t) = I(u+t\varphi) = \int_a^b f(x, u(x)+t\varphi(x), u'(x)+t\varphi'(x))\,dx$$

then $\Phi(t)$ must take a min. at $t=0$:

$$\Phi(t) \geq 0 \quad \Rightarrow \quad \frac{d}{dt}\Phi(t)\Big|_{t=0} = 0 .$$

If $f \in C^2([a,b]\times\mathbb{R}\times\mathbb{R})$, $f = f(x,y,z)$, then

$$\Phi'(0) = \int_a^b (\varphi f_y + \varphi' f_z)\,dx = 0.$$

If we assume $\inf\{I(u) : u \in X\}$ admits a minimiser $u \in X \cap C^2[a,b]$, then integration by parts implies

$$\int_a^b \varphi\left(f_y - \frac{d}{dx} f_z\right)dx = 0 .$$

As this holds for all φ, we conclude

$$\frac{d}{dx} f_z - f_y = 0$$

or in the expanded form

$$f_{zz}\cdot u'' + f_{zy}\cdot u' + f_{zx} = f_y .$$

NB: "regular" variational integrals are ones for which $f_{zz} > 0$.

§ The Direct Method : Outline

- Fix a class $\mathcal{E}$ of "admissible functions" with a suitable topology τ.
- Let $\mathcal{F} : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ be a given functional.

We need to show:

(1) $\mathcal{F}$ is well defined on $\mathcal{E}$ and bounded from below, i.e. $\inf \mathcal{F} > -\infty$.

From this we can take a minimising sequence $(u_j) \subset \mathcal{E}$ s.t. $\lim_{j\to\infty} \mathcal{F}(u_j) = \inf_{\mathcal{E}} \mathcal{F}$.

(2) (u_j) admits a converging subsequence (u_{j_k}), i.e. $u_{j_k} \xrightarrow{\tau} u_0 \in \mathcal{E}$.

(To do this use a compactness condition for the existence and closedness condition to ensure $u_0 \in \mathcal{E}$)

(3) $\mathcal{F}$ is sequentially lower semi-continuous w.r.t. τ, i.e.

$$\mathcal{F}(u_0) \leq \liminf_{j\to\infty} \mathcal{F}(u_j), \text{ whenever } u_j \xrightarrow{\tau} u_0 .$$

Conclusion:

$$\inf_{\mathcal{E}} \mathcal{F} \leq \mathcal{F}(u_0) \leq \liminf_{j\to\infty} \mathcal{F}(u_j) \leq \inf_{\mathcal{E}} \mathcal{F}$$

$$\Rightarrow \mathcal{F}(u_0) = \inf_{\mathcal{E}} \mathcal{F} . \quad \blacksquare$$

Remarks: There are some conflicting compatibility requirements:

→ For semicontinuity one prefers a relatively strong topology.

→ The opposite is true for compactness: The weaker the topology the easier it is for sequences to converge.

The direct method gives a very general existence result where solutions exist in a suitably general class in which continuity & differentiability may not be given.