EXISTENCE : DIRECT METHODS

§ The model case: Dirichlet integrand

<u>Theorem</u>. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $u_0 \in W^{1,2}(\Omega)$. Then

$$\inf \left\{ I(v) = \frac{1}{2}\int_\Omega |Dv|^2 dx : v \in u_0 + W_0^{1,2}(\Omega) \right\} = m$$

has one and only one solution $u \in u_0 + W_0^{1,2}(\Omega)$. Furthermore, u satisfies

$$\int_\Omega Du \cdot D\varphi \, dx = 0 \quad , \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Proof of existence

1. (compactness) As $u_0 \in u_0 + W_0^{1,2}(\Omega)$ we have

$$0 \le m \le I(u_0) < +\infty.$$

Now let $(u_j) \subset u_0 + W_0^{1,2}(\Omega)$ be a minimising sequence of (0), i.e. $I(u_j) \to \inf I = m$ as $j \to \infty$.

By Poincaré inequality, $\exists\, c_1, c_2 > 0$ s.t.

$$\sqrt{2I(u_j)} = \|Du_j\|_{L^2} \geq c_1 \|u_j\|_{W^{1,2}} - c_2 \|u_0\|_{W^{1,2}}.$$

As (u_j) is a minimising sequence and $m < +\infty$ we deduce that there exists $c_3 > 0$ s.t.

$$\|u_j\|_{W^{1,2}} \le c_3.$$

Therefore $\exists\, u \in u_0 + W_0^{1,2}(\Omega)$ and subsequence (u_{j_k}) s.t.

$$u_{j_k} \to u \text{ in } W^{1,2} \text{ as } j_k \to \infty.$$

2. (lower semicontinuity) [This is independent of (u_j) being a min. sequence].

First note that

$$|Du_j|^2 = |Du|^2 + 2\langle Du, Du_j - Du\rangle + |Du_j - Du|^2$$
$$\geq |Du|^2 + 2\langle Du, Du_j - Du\rangle$$

This implies that

$$I(u_j) \geq I(u) + \int_\Omega \langle Du, Du_j - Du\rangle \, dx$$

and as $Du \in L^2$ and $Du_j \to Du$ in L^2 we get

$$\lim_{j\to\infty} \int_\Omega \langle Du, Du_j - Du\rangle \, dx = 0.$$

Therefore,

$$\liminf_{j\to\infty} I(u_j) \geq I(u)$$

and so

$$m \le I(u) \le \liminf I(u_{j_k}) \le m. \quad \blacksquare$$

Proof of uniqueness

Assume there exists $u, v \in u_0 + W_0^{1,2}(\Omega)$ s.t. $I(u) = I(v) = m$.

By convexity,

$$0 \le m \le I\left(\frac{u+v}{2}\right) \le \frac{1}{2} I(u) + \frac{1}{2} I(v) = m.$$

This implies $\frac{1}{2}|Du|^2 + \frac{1}{2}|Dv|^2 - \left|\frac{Du+Dv}{2}\right|^2 = 0$ a.e. in Ω.

Then by strict convexity of $\xi \mapsto |\xi|^2$ we obtain $Du = Dv$ a.e. and as $u = v$ on $\partial\Omega$ we conclude $u \equiv v$ in Ω.

§ A general existence theorem

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $f \in C^0(\bar\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $f = f(x, u, \xi)$, satisfy

(H1) $\xi \mapsto f(x, u, \xi)$ is convex $\forall (x, u) \in \bar\Omega \times \mathbb{R}$ [Convexity]

(H2) There exists $p > q \geq 1$ and $\alpha_1 > 0$ & $\alpha_2, \alpha_3 \in \mathbb{R}$ such that [Coercivity]

$$f(x, u, \xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3, \quad \forall (x, u, \xi) \in \bar\Omega \times \mathbb{R} \times \mathbb{R}^n$$

then

$$\inf \left\{ I(u) = \int_\Omega f(x, u, \nabla u)\, dx : u \in u_0 + W_0^{1,2}(\Omega) \right\},$$

where $u_0 \in W^{1,2}(\Omega)$ with $I(u_0) < +\infty$, has a unique minimiser $u \in u_0 + W_0^{1,p}(\Omega)$.

Furthermore, if $(u, \xi) \mapsto f(x, u, \xi)$ is strictly convex for all $x \in \bar\Omega$, then the minimiser is unique.

Remarks. It is possible to weaken the continuity of f since it is enough to require f being a Carathéodory function

(i.e. $x \mapsto f(x, u, \xi)$ is measurable $\forall (u, \xi)$

and $(u, \xi) \mapsto f(x, u, \xi)$ is continuous a.e. x)

e.g. The above theorem does not apply to functionals of the form

$$f(x, u, \xi) = f(\xi) = \sqrt{1 + |\xi|^2},$$

since f satisfies (H2) only when $p = 1$.

e.g. (Weierstrass)

$$\inf \left\{ I(u) = \int_0^1 f(x, u')\, dx : u \in X \right\} \qquad (W)$$

where $f(x, \xi) = x \xi^2$ and $X = \{ u \in W^{1,2}(0,1) : u(0) = 1, u(1) = 0 \}$

- (H2) satisfied only with $\alpha_1 = 0$
- $W^{1,2}(0,1) \subset C^{0,1/2}(0,1)$ and so must be continuous

Hence this e.g. shows (W) has no min. in X!