§ Characterisation of the trace relative to the Sobolev – Slobodeckiĭ spaces.

For $0<s<1$ we can consider the Slobodeckiĭ space

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega\times\Omega) \right\}$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy.$$

The space $W^{s,p}(\Omega)$ is sometimes referred to as the "fractional Sobolev space".

e.g. One can show

$$W^{1/2,2}(S^1) = \left\{ u \in L^2(S^1) : \sum_{n\in\mathbb{Z}} n|a_n|^2 < +\infty, \text{ where } a_n = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)e^{-in\theta}\,d\theta \right\}$$

__Theorem__ (Gagliardo, 1957)

Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz domain and $p>1$. Then there exists a bounded linear operator

$$T : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$$

s.t.

(i) $Tu = u|_{\partial\Omega}$ for $u \in C(\bar\Omega)\cap W^{1,p}(\Omega)$

(ii) For all $u\in W^{1,p}(\Omega)$ and $\psi \in C_c^1(\mathbb{R}^n)$ we have

$$\int_\Omega u\frac{\partial\psi}{\partial x_i}\,dx = -\int_\Omega \frac{\partial u}{\partial x_i}\psi\,dx + \int_{\partial\Omega}\psi\, T(u)\,\nu_i\,d\sigma,$$

for $i=1,\dots,n$, where ν denotes the unit outer normal to $\partial\Omega$.

Conversely, for any $v\in W^{1-1/p,p}(\partial\Omega)$ there exists $u\in W^{1,p}(\Omega)$ s.t. $Tu=v$ and

$$\|u\|_{W^{1,p}(\Omega)} \le C(n,p,\Omega)\|v\|_{W^{1-1/p,p}(\partial\Omega)}.$$

__Remark.__ This result follows from the below theorem for cubes, the truncation lemma, the invariance of $W^{s,p}(\Omega)$ under coordinate transformation and the PoU + "flattening out the boundary" argument used previously.

Lemma. (Hardy's inequality)

Let $f\in L^p(a,b)$ for $p>1$ and $-\infty<a<b<+\infty$.

Then

$$\int_a^b \left[\frac{1}{x-a}\int_a^x |f(s)|\,ds\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_a^b |f(x)|^p\,dx$$

and

$$\int_a^b \left[\frac{1}{b-x}\int_x^b |f(s)|\,ds\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_a^b |f(x)|^p\,dx .$$

Proof. We want to use integration by parts. To make the boundary term vanish, we first consider

$$f_\varepsilon(x) = \begin{cases} f(x), & x\ge a+\varepsilon \\ 0, & a<x<a+\varepsilon \end{cases}.$$

Then,

$$\int_a^b \frac{1}{(x-a)^p}\left[\int_a^x |f_\varepsilon(\xi)|\,d\xi\right]^p dx$$

$$= \frac{1}{(b-a)^{p-1}}\,\frac{1}{1-p}\left(\int_a^b |f_\varepsilon|\right)^p + \frac{p}{p-1}\int_a^b \frac{1}{(x-a)^{p-1}}\left(\int_a^x |f_\varepsilon|\right)^{p-1}|f_\varepsilon|\,dx$$

$$\le \frac{p}{p-1}\left[\int_a^b \frac{1}{(x-a)^p}\left(\int_a^x |f_\varepsilon|\right)^p dx\right]^{\frac{p-1}{p}}\left(\int_a^b |f_\varepsilon|^p\right)^{1/p}.$$

we can then divide through by the common term to get the desired result for f_ε.

Afterwhich, applying Fatou's lemma yields the result. ∎

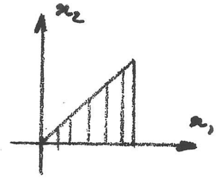
Lemma 1 (Gagliardo)

Let $p>1$ and $\Delta = \{(x_1,x_2)\in\mathbb{R}^2 : 0<x_1<1,\ 0<x_2<x_1\}$.

Then there exists $c>0$ s.t. for all $u\in W^{1,p}(\Delta)$ we have

$$\int_0^1 dt\int_0^t ds\left|\frac{u(t,t)-u(s,s)}{t-s}\right|^p \le c\,\|u\|^p_{W^{1,p}(\Delta)} .$$

Proof. Using the density up to the boundary result, it suffices to consider $u\in C^1(\bar\Delta)$.



Then we note that

$$\left|\frac{u(t,t)-u(s,s)}{t-s}\right|^p \le 2^{p-1}\left(\left|\frac{u(t,t)-u(t,s)}{t-s}\right|^p + \left|\frac{u(t,s)-u(s,s)}{t-s}\right|^p\right)$$

$$\le 2^{p-1}\left(\left[\frac{1}{t-s}\int_s^t\left|\frac{\partial u}{\partial x_2}(t,x_2)\right|dx_2\right]^p + \left[\frac{1}{t-s}\int_s^t\left|\frac{\partial u}{\partial x_1}(x_1,s)\right|dx_1\right]^p\right).$$

By integrating wrt $\{s: 0<s<t\}$ and then $\{t: 0<t<1\}$ we get

$$\int_0^1 dt\int_0^t ds\left|\frac{u(t,t)-u(s,s)}{t-s}\right|^p$$

$$\underset{\text{(Fubini)}}{\le} 2^{p-1}\int_0^1 dt\int_0^t ds\left[\frac{1}{t-\boxed{s}}\int_{\boxed{s}}^t\left|\frac{\partial u}{\partial x_2}(t,x_2)\right|dx_2\right]^p$$

$$+\, 2^{p-1}\int_0^1 ds\int_s^1 dt\left[\frac{1}{\boxed{t}-s}\int_s^{\boxed{t}}\left|\frac{\partial u}{\partial x_1}(x_1,s)\right|dx_1\right]^p$$

$$\underset{\text{(Hardy)}}{\le} 2^{p-1}\left(\frac{p}{p-1}\right)^p\left[\int_0^1 dt\int_0^t ds\left|\frac{\partial u(t,s)}{\partial x_2}\right|^p + \int_0^1 ds\int_s^1 dt\left|\frac{\partial u}{\partial x_1}(t,s)\right|^p\right]$$

$$= 2^{p-1}\left(\frac{p}{p-1}\right)^p\int_\Delta\left[\left|\frac{\partial u}{\partial x_1}(x_1,x_2)\right|^p + \left|\frac{\partial u}{\partial x_2}(x_1,x_2)\right|^p\right]dx_1\,dx_2 .$$ ∎

Lemma 2. Let $p>1$, $Q=(-1,1)^{n-1}$ for $n\geq 2$ and set

$$A_i(u) = \int_{(-1,1)^{n-2}} \widehat{dx_i} \int_{(-1,1)^2} dt\,ds \left|\frac{U_i(t)-U_i(s)}{t-s}\right|^p$$

where $\widehat{dx_i} = dx_1\cdots dx_{i-1}\,dx_{i+1}\cdots dx_{n-1}$

and $U_i(\tau) = u(x_1,\dots,x_{i-1},\underset{\substack{\uparrow\\ i\text{-th position}}}{\tau},x_{i+1},\dots,x_{n-1})$.

Then there exists $c>0$ s.t.

$$\|u\|^p_{W^{1-\frac{1}{p},p}(Q)} \leq c\left(\|u\|^p_{L^p(Q)} + \sum_{i=1}^{n} A_i(u)\right),$$

provided $u\in L^p(Q)$ and $A_i(u)<+\infty$ for $i=1,\dots,n-1$.

Proof. Let $u\in L^p(Q)$ and for $x,y\in Q$ denote

$$x^{(i)} = (x_1,\dots,x_i,y_{i+1},\dots,y_{n-1})$$

with $x^{(0)}=y$ and $x^{(n-1)}=x$.

Then we have

$$\int_Q\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n-2+p}}\,dx\,dy \leq c\sum_{i=1}^{n-1}\int_Q\int_Q \frac{|u(x^{(i)})-u(x^{(i-1)})|^p}{|x-y|^{n-2+p}}\,dx\,dy.$$

$$\uparrow\ \ n-2+p = n-1+\left(1-\tfrac{1}{p}\right)p$$

Each term on the RHS can be written in the form

$$\int_Q\int_Q \frac{|u(x^{(i)})-u(x^{(i-1)})|^p}{|x-y|^{n-2+p}}\,dx\,dy$$

$$= \int_{(-1,1)^n} dx_1\,dx_2\cdots dx_i\,dy_i\cdots dy_{n-1}\,|u(x^{(i)})-u(x^{(i-1)})|^p \times\left(\int_{(-1,1)^{n-2}} dx_{i-1}\cdots dx_{n-1}\,dy_1\cdots dy_{i-1}\,\frac{1}{|x-y|^{n-2+p}}\right)$$

We then:

Claim. There exists $k>0$ s.t. for $i=1,\dots,n-1$ we have

$$\int_{(-1,1)^{n-1}} dx_{i-1}\cdots dx_{n-1}\,dy_1\cdots dy_{i-1}\,\frac{1}{|x-y|^{n-2+p}} \leq \frac{k}{|x_i-y_i|^p}$$

Given this claim, we have

$$\int_Q\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n-2+p}}\,dx\,dy$$

$$\leq c\sum_{i=1}^{n-1}\int_{(-1,1)^n} dx_1\cdots dx_i\,dy_i\cdots dy_{n-1}\,\frac{|u(x^{(i)})-u(x^{(i-1)})|^p}{|x_i-y_i|^p}$$

$$= c\sum_{i=1}^{n-1}\underbrace{\int_{(-1,1)^{n-2}} dx_1\cdots dx_{i-1}\,dy_{i+1}\cdots dy_{n-1}\int_{(-1,1)^2} dx_i\,dy_i\,\frac{|u(x^{(i)})-u(x^{(i-1)})|^p}{|x_i-y_i|^p}}_{=A_i(u)}$$

(since $u(x^{(i)})-u(x^{(i-1)}) = U_i(x_i)-U_i(y_i)$)

∎

Now to prove the above claim, we check the estimate in the case $i=1$.

Hence to bound

$$\int_{(-1,1)^{n-2}} \frac{dx_2 \cdots dx_{n-1}}{|x-y|^{n-2+p}}$$

we first note that

$$\begin{aligned}|x-y|^{n-2+p} &= \left[(x_1-y_1)^2+\cdots+(x_{n-1}-y_{n-1})^2\right]^{\frac{n-2+p}{2}} \\ &= \left[\sum_{j=2}^{n-1}\left(|x_j-y_j|^2+\frac{|x_1-y_1|^2}{n-2}\right)\right]^{\frac{n-2+p}{2}} \\ &\geq \left[\sum_{j=2}^{n-1}\left(|x_j-y_j|^2+\frac{|x_1-y_1|^2}{n-2}\right)\right]^{n-2}|x_1-y_1|^{p-(n-2)}.\end{aligned}$$

Now using the AM-GM inequality(†) we estimate

$$\begin{aligned}&\int_{(-1,1)^{n-2}} \frac{dx_2 \cdots dx_{n-1}}{|x-y|^{n-2+p}} \\ &\leq \int_{(-1,1)^{n-2}} dx_2\cdots dx_{n-1} \frac{1}{\left[\sum_{j=2}^{n-2}\left((x_j-y_j)^2+\frac{|x_1-y_1|^2}{n-2}\right)\right]^{n-2}} \frac{1}{|x_1-y_1|^{p-(n-2)}} \\ &\underset{(\text{AM-GM})}{\leq} \frac{1}{|x_1-y_1|^{p-(n-2)}} \int_{(-1,1)^{n-2}} dx_2\cdots dx_{n-1} \prod_{j=2}^{n-1}\left[\frac{1}{(x_j-y_j)^2+\frac{(x_1-y_1)^2}{n-2}}\right] \\ &\leq \frac{1}{|x_1-y_1|^{p-(n-2)}} \prod_{j=2}^{n-1}\frac{C}{|x_1-y_1|} = \frac{C}{|x_1-y_1|^{p}},\end{aligned}$$

since $\displaystyle\int_{-1}^{1}\frac{dx_j}{(x_j-y_j)^2+\alpha^2} \leq \frac{C}{\alpha}$.

(†) i.e. $(z_1+\cdots+z_N)^N \geq z_1\cdots z_N$.

__Theorem.__ Let $Q = (-1,1)^n$ be an n-cube and let

$$Q_k = \{x : x_k = 1 \text{ and } |x_j| < 1 \text{ for } j \neq k,\ 1 \leq j \leq n\}$$

be the cube's k-face.

For $p > 1$ there exists a bounded linear map

$$tr : W^{1,p}(Q) \to W^{1-1/p,p}(Q_k)$$

s.t. $tr(u) = u|_{Q_k}$ for all $u \in C(\bar{Q}) \cap W^{1,p}(Q)$.

Proof. Let $u \in C^\infty(\bar{Q}) \cap W^{1,p}(Q)$. Then by Lemma 2 we have

$$\|u\|^p_{W^{1-1/p,p}(Q_k)} \leq c\Big(\|u\|^p_{L^p(Q_k)} + \sum_{\substack{1 \leq i \leq n \\ i \neq k}} A_i(u)^p\Big).$$

Assume that $i < k$, say, then denote

$$u_{ik}(t) = u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n).$$

$$\underset{\text{ith position}}{\uparrow} \qquad \underset{\text{k-th position}}{\uparrow}$$

By an abuse of notation, we write

$$\begin{aligned} u_{ik}(t) - u_{ik}(s) &= u(t,1) - u(s,1) \\ &= u(t,1) - u(t,s) \\ &\quad \left.\begin{aligned} &+ u(t,s) - u(t,t) \\ &+ u(t,t) - u(s,t) \end{aligned}\right\} \text{ use Lemma 1.} \\ &\quad + u(s,t) - u(s,1) \end{aligned}$$

For the other two terms we note that

$$|u(t,1) - u(t,s)| \leq \int_0^1 \left|\frac{\partial u(t,x)}{\partial x}\right| dx$$

and as before one can show the Hardy-type inequality in this case takes the form

$$\int_0^1 dt \left[\int_0^1 ds \frac{1}{(t-s)^p}\Big(\int_s^1 |f|\Big)^p\right] \leq c \int_0^1 dt \int_0^1 ds\, |f|^p,$$

where we set $f(x) = \dfrac{\partial u(t,x)}{\partial x}$.

In which case we can estimate

$$\int_{(-1,1)^{n-2}} dx_1 \dots \hat{dx_i} \dots \hat{dx_k} \dots dx_n \left(\int_{-1}^1\int_{-1}^1 \frac{|u_{ik}(t) - u_{ik}(s)|^p}{|t-s|^p} dt\,ds\right)$$

$$\leq c \int_{(-1,1)^{n-2}} dx_1 \dots \hat{dx_i} \dots \hat{dx_k} \dots dx_n$$

$$\times \left(\int_{-1}^1 \int_{-1}^1 \left[\left|\frac{\partial u}{\partial x_k}\right|^2 + \left|\frac{\partial u}{\partial x_i}\right|^2\right] dx_i\, dx_k\right)$$

and by the L^p-version of the trace theorem we also have

$$\|u\|_{L^p(Q_k)} \leq c\|u\|_{W^{1,p}(Q)}. \qquad \blacksquare$$

§ Invariance of $W^{s,p}$-functions relative to a coordinate change.

Suppose there is a coordinate transformation between domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ given by

$$x_j = x_j(t_1, \dots, t_n) \qquad j = 1, \dots, n.$$

If the one to one transformation is bi-Lipschitz, then there exists constants $0 < k_1 < k_2$ s.t.

$$k_1 |t - \tau| \le |x(t) - x(\tau)| \le k_2 |t - \tau| \qquad \text{——} (*)$$

for all $t, \tau \in \tilde{\Omega}$.

Lemma. For a bi-Lipschitz coordinate transformation $x = x(t) : \tilde{\Omega} \to \Omega$ we have

$$c_1 \int_{\Omega\times\Omega} |f(x) - f(y)|^p \frac{dx\,dy}{|x-y|^\mu} \le \int_{\tilde{\Omega}\times\tilde{\Omega}} |f\circ x(t) - f\circ x(\tau)|^p \frac{dt\,d\tau}{|t-\tau|^\mu}$$

$$\le c_2 \int_{\Omega\times\Omega} |f(x) - f(y)|^p \frac{dx\,dy}{|x-y|^\mu}$$

for some positive constants c_1 and c_2 independent of f.

Proof. We have

$$\int_{\tilde{\Omega}\times\tilde{\Omega}} |f\circ x(t) - f\circ x(\tau)|^p \frac{dt\,d\tau}{|t-\tau|^\mu}$$

$$= \int_{\Omega\times\Omega} |f(x) - f(y)|^p \Big/ \left|\frac{\partial(t_1, \dots, t_n)}{\partial(x_1, \dots, x_n)}\right| \left|\frac{\partial(\tau_1, \dots, \tau_n)}{\partial(y_1, \dots, y_n)}\right| \frac{dx\,dy}{|t(x) - t(y)|^\mu}$$

where the Jacobian $\dfrac{\partial(t_1, \dots, t_n)}{\partial(x_1, \dots, x_n)}$ has bounds determined by $(*)$.

Likewise,

$$\int_{\Omega\times\Omega} |f(x) - f(y)|^p \frac{dx\,dy}{|x-y|^\mu}$$

$$= \int_{\tilde{\Omega}\times\tilde{\Omega}} |f\circ x(t) - f\circ x(\tau)|^p \Big/ \left|\frac{\partial(x_1, \dots, x_n)}{\partial(t_1, \dots, t_n)}\right| \left|\frac{\partial(y_1, \dots, y_n)}{\partial(\tau_1, \dots, \tau_n)}\right| \frac{dt\,d\tau}{|x(t) - y(\tau)|^\mu}$$

The desired bounds now follow by the bi-Lipschitz property. $\blacksquare$

Lemma. The space $W^{s,p}(\Omega)$, $0 < s < 1$, is invariant with respect to any bi-Lipschitz transformation, i.e. if $x = x(t) : \tilde{\Omega} \to \Omega$ is bi-Lipschitz, then there exists $C > 0$ s.t.

$$\frac{1}{C} \|f\|_{W^{s,p}(\Omega)} \le \|f\circ x\|_{W^{s,p}(\tilde{\Omega})} \le C \|f\|_{W^{s,p}(\Omega)}.$$

Now, a truncation lemma near $\partial\Omega$.

Lemma 5.3. *Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1,+\infty)$. Let us consider $u \in W^{s,p}(\Omega)$ and $\psi \in C^{0,1}(\Omega)$, $0 \le \psi \le 1$. Then $\psi u \in W^{s,p}(\Omega)$ and*

$$\|\psi\, u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{s,p}(\Omega)}, \tag{5.6}$$

where $C = C(n,p,s,\Omega)$.

Proof. It is clear that $\|\psi\, u\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)}$ since $|\psi| \le 1$. Furthermore, adding and subtracting the factor $\psi(x)u(y)$, we get

$$\begin{aligned}
\int_\Omega\int_\Omega & \frac{|\psi(x)\,u(x)-\psi(y)\,u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\le 2^{p-1}\left(\int_\Omega\int_\Omega \frac{|\psi(x)\,u(x)-\psi(x)\,u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \right.\\
&\quad \left. + \int_\Omega\int_\Omega \frac{|\psi(x)\,u(y)-\psi(y)\,u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right) \\
&\le 2^{p-1}\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \right. \qquad (5.7)\\
&\quad \left. + \int_\Omega\int_\Omega \frac{|u(x)|^p\,|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right).
\end{aligned}$$

Since ψ belongs to $C^{0,1}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{|u(x)|^p\,|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}\,dx\,dy &\le \Lambda^p\int_\Omega\int_{\Omega\cap|x-y|\le 1} \frac{|u(x)|^p\,|x-y|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad + \int_\Omega\int_{\Omega\cap|x-y|\ge 1} \frac{|u(x)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\le \tilde{C}\|u\|^p_{L^p(\Omega)}, \qquad (5.8)
\end{aligned}$$

where Λ denotes the Lipschitz constant of ψ and $\tilde{C}$ is a positive constant depending on n, p and s. Note that the last inequality follows from the fact that the kernel $|x-y|^{-n+(1-s)p}$ is summable with respect to y if $|x-y| \le 1$ since $n+(s-1)p < n$ and, on the other hand, the kernel $|x-y|^{-n-sp}$ is summable when $|x-y| \ge 1$ since $n+sp > n$. Finally, combining (5.7) with (5.8), we obtain estimate (5.6). □