REGULARITY RESULTS

§ Weakly harmonic functions

Lemma (Caccioppoli)

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = 0$ in Ω.
Then for each $x_0 \in \Omega$, $a \in \mathbb{R}$, $0 < r < R < \mathrm{dist}(x_0, \partial\Omega)$ we have

$$\int_{B_r(x_0)} |Du|^2 \, dx \le \frac{16}{(R-r)^2} \int_{B_R(x_0)} |u-a|^2 \, dx .$$

Proof. Define $\eta \in C_c^\infty(\Omega)$ s.t. $0 \le \eta \le 1$, $\begin{cases} \eta = 1 & \text{on } B_r(x_0) \\ \eta = 0 & \text{on } B_R(x_0) \setminus B_r(x_0) \end{cases}$

and $|D\eta| < \dfrac{2}{R-r}$.

By the weak formulation

$$\int_\Omega Du \, D\varphi \, dx = 0 , \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Now choose $\varphi = \eta^2(u-a)$. Then

$$0 = \int_\Omega Du \cdot D\varphi \, dx = \int_\Omega Du \left(2\eta D\eta (u-a) + \eta^2 Du \right) dx$$

so that

$$\int_\Omega \eta^2 |Du|^2 dx \le \int |\eta Du| \cdot 2|D\eta| |u-a| \, dx$$
$$\le \| \eta Du \|_2 \, \| 2|D\eta| |u-a| \|_2$$

$$\Rightarrow \quad \| \eta Du \|_2 \le \| 2|D\eta| |u-a| \|_2$$

$$\Rightarrow \quad \int_{B_r(x_0)} |Du|^2 dx \le \frac{16}{(R-r)^2} \int_{B_R} |u-a|^2 dx . \quad \blacksquare$$

Lemma (Weyl)

If $u \in W^{1,2}(\Omega)$ is a weak solution of $\Delta u = 0$ in Ω, then $u \in C^\infty_{loc}(\Omega)$.

Proof. Using a difference quotient argument (cf. Problem Sheet #21), there exists $C = C(n) > 0$ s.t.

$$\int_{B_{R/2}} |D^2 u|^2 dx \le \frac{C}{R^2} \int_{B_R(x_0)} |Du|^2 dx , \quad B_R(x_0) \subset\subset \Omega .$$

As higher order derivatives of u are solutions of $\Delta u = 0$ also, an iteration argument, together with the Caccioppoli inequality, implies that

$$\int_{B_{R/2}(x_0)} |D^k u|^2 dx \le \frac{C}{R^{2k}} \int_{B_R(x_0)} |u|^2 dx$$

for all $B_R(x_0) \subset\subset \Omega$ and each integer $k \in \mathbb{N}$.

Now by the Sobolev embeddings, $W^{k+1,2}(\Omega') \subseteq C^{k+1-n/2}(\Omega')$, $\Omega' \subset\subset \Omega$.
Thus we have $u \in C^\infty_{loc}(\Omega) = \bigcap_{k=0}^\infty C^k_{loc}(\Omega)$. $\blacksquare$

§ Elliptic regularity: De Giorgi's result

Definition (De Giorgi class)

Let $DG(\Omega)$ denote the class of functions $u \in W^{1,2}(\Omega)$ for which there exists $c>0$ s.t. for $x_0 \in \Omega$, $k \in \mathbb{R}$ and $0<r<R<\mathrm{dist}(x_0,\partial\Omega)$ we have

$$\int_{A(k)\cap B_r(x_0)} |Du|^2\,dx \le \frac{c}{(R-r)^2}\int_{A(k)\cap B_R(x_0)} (u-k)^2\,dx,$$

where $A(k) = \{x\in\Omega : u(x)>k\}$.

Theorem I (De Giorgi '57)

If $u, -u \in DG(\Omega)$, then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $0<\alpha<1$.

Moreover, $\omega(r) < \left(\frac{r}{R}\right)^{\alpha}\omega(R)$ where $\omega(r) = \sup_{B_r} u - \inf_{B_r} u$.

Theorem II (De Giorgi '57)

Let $u \in W^{1,2}(\Omega)$ be a weak solution to

$$\mathrm{div}(a^{ij}D_j u) = 0$$

under the assumption that

$$|a^{ij}(x)| \le L, \quad a^{ij}(x)\xi_i\xi_j \ge \ell|\xi|^2$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^n$.

Then $u\in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(n, L/\ell) > 0$.

NB: The real achievement here lies in the fact that a^{ij} are "a priori" only bounded measurable functions.

Lemma (Caccioppoli 1934/50)

Suppose the functions $\varphi=\varphi(r)$ and $\psi=\psi(r)>0$ satisfy the inequality

$$\psi(r)\varphi(r) - \lambda\int_0^r \varphi(t)^2\,dt \ge 0,$$

for all $0<r\le R$, where $\lambda>0$ is a constant.

Then for all $0<r<R$ we have

$$\int_0^r \varphi(t)^2\,dt \le \frac{1}{\lambda^2}\,\frac{1}{(R-r)^2}\int_0^R \psi(t)^2\,dt.$$

Proof. Set $F(r) = \int_0^r \varphi(t)^2\,dt$ and assume wlog $F(R)>0$.

Now re-write the hypothesis as

$$\frac{\varphi(r)}{F(r)} \ge \underset{\substack{\uparrow \\ (\text{as } \psi>0)}}{\frac{\lambda}{\psi(r)}} > 0.$$

Then

$$\frac{F'(r)}{F(r)^2} = \frac{\varphi(r)^2}{F(r)^2} \ge \frac{\lambda^2}{\psi(r)^2}.$$

So for any $0<r_1<r_2\le R$ we have

$$\frac{1}{\lambda^2}\left[\frac{1}{F(r_1)} - \frac{1}{F(r_2)}\right] \ge \int_{r_1}^{r_2}\frac{dt}{\psi(t)^2} \underset{\substack{\uparrow \\ \text{Cauchy-Schwarz}}}{\ge} \frac{(r_2-r_1)^2}{\int_{r_1}^{r_2}\psi(t)^2\,dt}.$$

using the fact that $F(R) > 0$ then implies

$$\frac{1}{\lambda^2}\frac{1}{F(r)} \geq \frac{(R-r)^2}{\int_0^R \psi(t)^2\,dt}, \quad 0 < r < R.$$

$$\Rightarrow \quad F(r) \leq \frac{1}{\lambda^2}\frac{1}{(R-r)^2}\int_0^R \psi(t)^2\,dt. \quad \blacksquare$$

Proof of Theorem II.

Fix $y \in \Omega$ and $k \in \mathbb{R}$. Set $\varphi(x) = \begin{cases} u(x) - k, & x \in A(k) \\ 0, & x \in \Omega \setminus A(k). \end{cases}$

Then $\varphi \in W^{1,2}(\Omega)$ also.

Now take $0 < r < \mathrm{dist}(y, \partial\Omega)$ and a smooth radial function η such that $\eta = 0$ on $[r, \infty)$.

Then set $\chi(x) = \eta(|x|)$.

We then have

$$\chi\varphi \in W^{1,2}(\Omega) \text{ and } \chi\varphi = 0 \text{ on } \Omega \setminus B_r(y).$$

From the given weak solution we have

$$\int_{B_r(y)} a^{ij} D_j u \, D_i(\chi\varphi)\,dx = 0.$$

By the co-area formula we find that

$$0 = \int_0^r d\rho \Big(\int_{\partial B_\rho(y)} a^{ij} D_j u \underbrace{D_i(\chi\varphi)}\,d\sigma(x) \Big)$$

$$= \chi D_i\varphi + \eta'(|x|) D_i|x| \, \varphi(x)$$

$$D_i|x| = \frac{x^i}{|x|} = \nu_i$$

$$= \int_0^r d\rho \Big[\eta'(\rho) \int_{\partial B_\rho(y)} a^{ij} D_j u \, \varphi \, \nu_i \, d\sigma(x) + \underbrace{\eta(\rho) \int_{\partial B_\rho(y)} a^{ij} D_j u \, D_i\varphi \, d\sigma(x)}_{\text{(integration by parts)}} \Big]$$

$$= \int_0^r d\rho \, \eta'(\rho) \Big[\int_{\partial B_\rho(y)} a^{ij} D_j u \, \varphi \, \nu_i \, d\sigma(x) - \int_0^\rho ds \Big(\int_{\partial B_s(y)} a^{ij} D_j u \, D_i \varphi \, d\sigma(x) \Big) \Big]$$

since

$$\int_0^r d\rho \, \eta(\rho) F'(\rho) = -\int_0^r d\rho \, \eta'(\rho) F(\rho),$$

where

$$F(\rho) = \int_0^\rho ds \int_{\partial B_s(y)} f \, d\sigma(x) \quad \text{so that} \quad F'(\rho) = \int_{\partial B_\rho(y)} f \, d\sigma(x).$$

Using the fact that η is arbitrary and the definition of φ, we conclude that

$$\int_{A(k)\cap\partial B_\rho(y)} a^{ij} \partial_j u \,(u-k)\, \nu_i \, d\sigma(x) = \int_{A(k)\cap B_\rho(y)} a^{ij} \partial_j u \, \partial_i u \, dx$$

for a.e. $0<\rho<\mathrm{dist}(y,\partial\Omega)$.

Then by the ellipticity condition we find that

$$\ell \int_{A(k)\cap B_\rho(y)} |Du|^2 dx \le L \int_{A(k)\cap\partial B_\rho(y)} |u-k|\,|Du| \, d\sigma(x). \qquad \text{---} \ (*)$$

Now set

$$\begin{cases} \psi_1(\rho) = \displaystyle\int_{A(k)\cap\partial B_\rho(y)} (u-k)^2 \, d\sigma(x) \\ \psi_2(\rho) = \displaystyle\int_{A(k)\cap\partial B_\rho(y)} |Du|^2 \, d\sigma(x). \end{cases}$$

Then by $(*)$ and Cauchy-Schwartz we get

$$\int_0^\rho \psi_2(t)\,dt \le \frac{L}{\ell} \sqrt{\psi_1(\rho)\psi_2(\rho)}.$$

So by Caccioppoli's lemma we conclude

$$\int_0^{\rho_1} \psi_2 \, dt \le \left(\frac{L}{\ell}\right)^2 \frac{1}{(\rho_2-\rho_1)^2} \int_0^{\rho_2} \psi_1 \, dt$$

for $0<\rho_1<\rho_2<\mathrm{dist}(y,\partial\Omega)$.

The same holds for $-u$. $\blacksquare$

On the differentiability and the analyticity of extremals of regular multiple integrals[‡][†]

Memoir by Ennio De Giorgi[*]

Summary. We study the extremals of some regular multiple integrals: assuming that the first order derivatives exist and are square-summable, we show that they are Hölder continuous. It follows that the extremals are infinitely differentiable and real analytic.

In this paper I deal with the differentiability properties of the extremals of regular multiple integrals, and in particular with their analyticity. This topic has been the object of several investigations of both Italian and foreign mathematicians, and hence it appears to be quite difficult to give a complete bibliographical account. For this reason, we shall limit ourselves to quote a few papers where the reader can find further information. Let us only mention the results by Hopf [3][1], Stampacchia [9], Morrey [6], who give differentiability and analyticity results for less and less regular extremals. In particular, in [3] the existence and Hölder continuity of second order derivatives, in [9] of first order derivatives, in [6] the existence and continuity of first order derivatives is assumed. The results obtained by Stampacchia in [9] belong to another direction of research. He moves from existence theorems obtained by the direct methods of the calculus of variations, where solutions are found in very wide classes of functions, and studies the properties of these (a priori very little regular) solutions. Among other results, he proves the existence of square-summable second order derivatives, satisfying the Euler equation almost everywhere.

What was still missing, to my knowledge (with the exception of double integrals, see [2], [5], [7], [8], and some particular cases of multiple integral, as quadratic integrals, which give rise to linear Euler equations), were theorems which could bridge the gap between the results obtained in the first research line and those in the second, i.e., theorems ensuring that the solutions obtained by direct methods and studied in [9] satisfy the conditions required in [6]. The aim of this paper is to show a first theorem in this direction (see Theorem III[2]). Its proof is based on the study of some functions (characterized by certain integral inequalities) which are Hölder continuous (see Theorem I). Among the

‡Editor's note: translation into English of the paper "Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari", published in Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat., **(3) 3** (1957), 25–43.

†Work performed at the Istituto Nazionale per le Applicazioni del Calcolo

*Presented by the Socio nazionale non residente MAURO PICONE in the meeting of April 25, 1957

1 Numbers in brackets refer to the Bibliography at the end of this paper.

2 This theorem has been presented in a talk at the U.M.I. meeting held in Pavia from 6 to 11 October, 1955 and also in the preliminary Note [1].

intermediate results, let us mention Theorem II, because it could be interesting also in other problems concerning elliptic partial differential equations.

This research has been suggested to me by some conversations with Prof. G. Stampacchia. I am grateful to him for the information and the advice he gave me, that have been very useful.

1. – Let us consider an open subset E of the Euclidean r-dimensional space S_r, and let us denote by $\mathcal{U}^{(2)}(E)$ the class of the functions $w(x)$ almost continuous in E which satisfy the following conditions:

1^{st}) *$w(x)$ is absolutely continuous on almost all segments contained in E parallel to the coordinate axes.*

2^{nd}) *$w(x)$ and its first partial derivatives are square-summable in every compact subset of E.*

Given a positive number γ, we denote by $\mathcal{B}(E;\gamma)$ the class of the functions $w(x)$ which, beside conditions 1^{st}) and 2^{nd}), satisfy also the following

3^{rd}) *Given $y \in E$ [3] (whose distance from $S_r \setminus E$ is denoted by $\delta(y)$) and given three numbers k, ϱ_1, ϱ_2 such that $0 < \varrho_1 < \varrho_2 < \delta(y)$ the inequalities*

$$\frac{\gamma}{(\varrho_2-\varrho_1)^2}\int_{A(k)\cap I(\varrho_2;y)} (w(x)-k)^2\,dx_1\dots dx_r \geq \geq \int_{A(k)\cap I(\varrho_1;y)} |\operatorname{grad} w|^2\,dx_1\dots dx_r, \tag{1}$$

$$\frac{\gamma}{(\varrho_2-\varrho_1)^2}\int_{B(k)\cap I(\varrho_2;y)} (w(x)-k)^2\,dx_1\dots dx_r \geq \geq \int_{B(k)\cap I(\varrho_1;y)} |\operatorname{grad} w|^2\,dx_1\dots dx_r, \tag{1'}$$

hold, where $I(\varrho;y)$ denotes the ball with centre y and radius ϱ, $A(k)$ denotes the subset of E where $w(x) > k$, and $B(k)$ the subset of E where $w(x) < k$.

A first property of the class $\mathcal{B}(E;\gamma)$ just defined, which will be useful later, is given in the following

LEMMA I. – *Let a sequence of functions*

$$w_1(x), \dots, w_n(x), \dots \tag{2}$$

in $\mathcal{B}(E;\gamma)$ be given, with $|w_n(x)|^2$ summable in E for every n. If (2) converges in quadratic mean in E to a function $w(x)$, i.e.

$$\lim_{n\to\infty}\int_E \left[w_n(x)-w(x)\right]^2 dx_1\dots dx_r = 0, \tag{3}$$

then $w(x)$ belongs to $\mathcal{B}(E;\gamma)$.

3 The symbol $y \in E$ means: y belongs to E; the symbol $E \subset L$ means: E is contained in L.

and then, by (57), (61) we deduce

$$\text{meas}\, A(\mu_1 - 2\eta\omega; 2\varrho) < \theta\left(\frac{1}{2}\right)\varrho^r < \theta\left(\frac{1}{2}\right)(2\varrho)^r. \tag{62}$$

Recalling that μ_1 is the true least upper bound of $w(x)$ in $I(4\varrho)$, by (62) we have

$$\int_{A(\mu_1-2\eta\omega;2\varrho)} (w(x) - \mu_1 + 2\eta\omega)^2\, dx_1 \ldots dx_r < (2\varrho)^r(2\eta\omega)^2\theta\left(\frac{1}{2}\right) \tag{63}$$

and then, by Lemma IV

$$\text{meas}\, A(\mu_1 - \eta\omega, \varrho) = 0, \tag{64}$$

i.e., the true least upper bound of $w(x)$ in $I(\varrho)$ does not exceed $(\mu_1 - \eta\omega)$ and (49) holds true. By Remark I, if we assume (51') instead of (51), we reach the same conclusion.

3. – The lemmas proved in Section 2 allow us to prove the following result.

THEOREM I. – *Each function $w(x) \in \mathcal{B}(E;\gamma)$ is uniformly Hölder continuous in every compact subset of E*[6].

Proof. From Remark II and Lemma V in Section 2 we deduce that, for every $y \in E$, the oscillation of $w(x)$ in $I(\varrho; y)$ is infinitesimal as $\varrho \to 0$; as a consequence, for every $y \in E$ the limit

$$\bar{w}(y) = \lim_{\varrho\to 0}\left[\text{meas}\, I(\varrho;y)\right]^{-1}\int_{I(\varrho;y)} w(x)\, dx_1 \ldots dx_r \tag{1}$$

exists, and $\bar{w}(x)$ is continuous in E.

Let us now set

$$\alpha = -\log_4(1-\eta). \tag{2}$$

Here η is the constant in Lemma V which, by (58), (57), (38), (37), (33), (12) in Section 2 is independent of y in E. Fix a compact set $C \subset E$ and a positive number p less than or equal to the distance of C from the boundary of E, and consider the set L whose elements are the numbers

$$\frac{2|\bar{w}(x) - \bar{w}(y)|4^\alpha}{p^\alpha} \tag{3}$$

for

$$0 \le |x - y| \le p, \qquad y \in C, \tag{4}$$

where $|x - y|$ is the distance between x and y.

[6] As usual, this must be understood in the sense of integration theory, i.e., either $w(x)$ itself is Hölder continuous, or there exists a Hölder continuous function coinciding almost everywhere with $w(x)$; analogous remarks are often understood in this paper (see [1] footnote [2]).

Since $\bar{w}(x)$ is continuous in E, the set L has an absolute maximum, which we shall denote by τ and then, fixed $y \in C$ we have, with the same notation as in Section 2 and in particular in Lemma V,

$$\text{osc}\,(w;\varrho) \le \tau\varrho^\alpha \quad \text{for} \quad \frac{p}{4} \le \varrho \le p. \tag{5}$$

Notice that for every positive $\varrho < \dfrac{p}{4}$ there is an integer m such that

$$\frac{p}{4} \le 4^m\varrho < p \tag{6}$$

and then, by Lemma V and (2), (5), (6), we have

$$\text{osc}\,(w;\varrho) \le (1-\eta)^m \text{osc}\,(w; 4^m\varrho) \le \tau\varrho^\alpha. \tag{7}$$

By the arbitrariness of C and y the proof is complete.

THEOREM II. – *Let r^2 functions $a_{hl}(x)$, almost continuous in the open set $E \subset S_r$, be given. Assume that $a_{hl}(x) = a_{lh}(x)$ and that two positive numbers τ_1, τ_2 exist, such that the inequalities*

$$\tau_1|\lambda|^2 \le \sum_{h,l}^{1,r} a_{hl}(x)\lambda_h\lambda_l \le \tau_2|\lambda|^2 \tag{8}$$

hold for every $x \in E$ and for every vector $\lambda \equiv (\lambda_1, \ldots, \lambda_r)$. Let also $w(x)$ be a function in $\mathcal{U}^{(2)}(E)$ such that, for every compact set $C \subset E$ and for every function $g(x) \in \mathcal{U}^{(2)}(E)$, which vanishes in $(E \setminus C)$, we have

$$\sum_{h,l}^{1,r}\int_E \frac{\partial g}{\partial x_h} a_{hl}(x)\frac{\partial w}{\partial x_l}\, dx_1 \ldots dx_r = 0. \tag{9}$$

Then, $w(x)$ belongs to $\mathcal{B}(E;\gamma)$, with $\gamma = \left(\frac{\tau_2}{\tau_1}\right)^2$, and then it is Hölder continuous in E by Theorem I.

Proof. Fix a point $y \in E$ and a real number k, and set, with the same notation as in Section 1,

$$\begin{aligned} \varphi(x) &\equiv w(x) - k \qquad &&\text{if} \quad x \in A(k) \\ \varphi(x) &\equiv 0 &&\text{if} \quad x \in (E \setminus A(k)). \end{aligned} \tag{10}$$

It is easily checked that $\varphi(x)$ belongs to $\mathcal{U}^{(2)}(E)$ as well. Take a positive number $p < \delta(y)$ and a function $u(t)$ depending on t, continuous with its first derivative $u'(t)$ in the interval $[0, +\infty]$, and vanishing in $[p, +\infty]$, and set

$$f(x_1, \ldots, x_r) = u(\sqrt{(x_1-y_1)^2 + \ldots + (x_r-y_r)^2}); \tag{11}$$

the function $\varphi(x)\cdot f(x)$ belongs to $\mathcal{U}^{(2)}(E)$ and vanishes in $[E \setminus I(p;y)]$; from (9) it follows

$$\sum_{h,l}^{1,r}\int_E \frac{\partial(\varphi\cdot f)}{\partial x_h} a_{hl}(x)\frac{\partial w}{\partial x_l}\, dx_1 \ldots dx_r = 0. \tag{12}$$

Since $f(x)$ vanishes identically in $[E \setminus I(p;y)]$, from (11), (12) we deduce

$$\sum_{h,l}^{1,r} \int_0^p d\varrho \left[u'(\varrho) \int_{\mathcal{F}I(\varrho;y)} n_h a_{hl}(x) \varphi(x) \frac{\partial w}{\partial x_l} \, d\mu_{r-1} + \right. \tag{13}$$
$$\left. + u(\varrho) \int_{\mathcal{F}I(\varrho;y)} a_{hl}(x) \frac{\partial \varphi}{\partial x_h} \frac{\partial w}{\partial x_l} \, d\mu_{r-1} \right] = 0,$$

where $n_1, \ldots, n_r$ are the components of the outward pointing unit normal to $\mathcal{F}I(\varrho;y)$ and $d\mu_{r-1}$ is the $(r-1)$-dimensional measure. From (13), integrating by parts, we deduce

$$\sum_{h,l}^{1,r} \int_0^p d\varrho u'(\varrho) \left[\int_{\mathcal{F}I(\varrho;y)} n_h a_{hl}(x) \varphi(x) \frac{\partial w}{\partial x_l} \, d\mu_{r-1} - \right. \tag{14}$$
$$\left. - \int_0^\varrho dt \int_{\mathcal{F}I(t;y)} a_{hl}(x) \frac{\partial \varphi}{\partial x_h} \frac{\partial w}{\partial x_l} \, d\mu_{r-1} \right] = 0.$$

By the arbitrariness of p and $u(t)$ we have, for almost every positive number $\varrho < \delta(y)$,

$$\int_{\mathcal{F}I(\varrho;y)} \sum_{h,l}^{1,r} n_h a_{hl} \varphi(x) \frac{\partial w}{\partial x_l} \, d\mu_{r-1} = \int_{I(\varrho;y)} \sum_{h,l}^{1,r} a_{hl} \frac{\partial \varphi}{\partial x_h} \frac{\partial w}{\partial x_l} \, dx_1 \ldots dx_r. \tag{15}$$

By (10), (15) we have then

$$\int_{A(k)\cap\mathcal{F}I(\varrho;y)} \sum_{h,l}^{1,r} n_h a_{hl}(x)(w(x)-k) \frac{\partial w}{\partial x_l} \, d\mu_{r-1} = \tag{16}$$
$$\int_{A(k)\cap I(\varrho;y)} \sum_{h,l}^{1,r} a_{hl}(x) \frac{\partial w}{\partial x_h} \frac{\partial w}{\partial x_l} \, dx_1 \ldots dx_r,$$

which by (8) implies

$$\tau_2 \int_{A(k)\cap\mathcal{F}I(\varrho;y)} (w(x)-k) |\operatorname{grad} w| \, d\mu_{r-1} \geq \tag{17}$$
$$\geq \tau_1 \int_{A(k)\cap I(\varrho;y)} |\operatorname{grad} w|^2 \, dx_1 \ldots dx_r.$$

Setting now

$$\psi_1(\varrho) = \int_{A(k)\cap\mathcal{F}I(\varrho;y)} (w(x)-k)^2 \, d\mu_{r-1},$$
$$\psi_2(\varrho) = \int_{A(k)\cap\mathcal{F}I(\varrho;y)} |\operatorname{grad} w|^2 d\mu_{r-1}, \tag{18}$$

from (17), taking into account Schwarz inequality, we deduce

$$\sqrt{\psi_1(\varrho)\psi_2(\varrho)} \geq \frac{\tau_1}{\tau_2} \int_0^\varrho \psi_2(t) dt. \tag{19}$$

Setting $\gamma = \left(\frac{\tau_2}{\tau_1}\right)^2$, from (19) and a Lemma due to Caccioppoli and Leray (see [4] on page 153), it follows

$$\int_0^{\varrho_1} \psi_2(\varrho) \, d\varrho \leq \frac{\gamma}{(\varrho_2 - \varrho_1)^2} \int_0^{\varrho_2} \psi_1(\varrho) \, d\varrho, \tag{20}$$

for $0 < \varrho_1 < \varrho_2 < \delta(y)$. From (18) and (20) inequality (1) in Section 1 immediately follows. Since whenever $w(x)$ satisfies the hypotheses of the statement, the same holds for $-w(x)$, it is easily checked that (1') in Section 1 holds as well, so that

$$w(x) \in \mathcal{B}(E;\gamma). \tag{21}$$

4. – We are now in a position to prove the announced analyticity theorem. To this aim, let us consider a function $f(p) \equiv f(p_1, \ldots, p_r)$, which we assume to be continuous in S_r together with its first and second order partial derivatives. Let us set

$$f_{hk}(p) = \frac{\partial^2 f}{\partial p_h \partial p_k}, \quad f_h(p) = \frac{\partial f}{\partial p_h} \qquad (\text{for } h, k = 1, \ldots, r) \tag{1}$$

and assume that there are two positive numbers μ_1 and μ_2 such that for every $p \in S_r$ and for every vector

$$\lambda \equiv (\lambda_1, \ldots, \lambda_r),$$

we have

$$\mu_1 |\lambda|^2 \leq \sum_{h,k}^{1,r} f_{hk}(p) \lambda_h \lambda_k \leq \mu_2 |\lambda|^2. \tag{2}$$

Given an open set $E \subset S_r$ and a function $u^*(x) \in \mathcal{U}^{(2)}(E)$, we say that $u^*(x)$ *is extremal in* E *for the integral functional*

$$\mathcal{I}[u] = \int f\left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_r}\right) dx_1 \ldots dx_r \tag{3}$$

if for every compact subset $C \subset E$ and for every function $g(x)$ which is continuous in E together with its first order derivatives and vanishes in $(E \setminus C)$ we have

$$\sum_{h=1}^{r} \int_E \frac{\partial g}{\partial x_h} f_h\left(\frac{\partial u^*}{\partial x_1}, \ldots, \frac{\partial u^*}{\partial x_r}\right) dx_1 \ldots dx_r = 0. \tag{4}$$