EXISTENCE OF WEAK SOLUTIONS

§ Lax-Milgram method

We first recall:

<u>Theorem (Riesz Representation)</u>

For any bounded linear functional F on a Hilbert space $\mathcal{H}$, there is a unique $f \in \mathcal{H}$ s.t.

$$F(x) = \langle x, f \rangle \quad \text{for all } x \in \mathcal{H}$$

and $\|F\| = \|f\|$.

We would like a slight generalisation of the Riesz rep. result.

<u>Def.</u> A bilinear form B on a Hilbert space $\mathcal{H}$ is <u>bounded</u> if there exists a constant $K > 0$ s.t.

$$|B(x,y)| \le K \|x\| \|y\| \quad \text{for all } x, y \in \mathcal{H} \qquad — (1)$$

and B is said to be <u>coercive</u> if there exists $\nu > 0$ s.t.

$$B(x,x) \ge \nu \|x\|^2 \quad \text{for all } x \in \mathcal{H}. \qquad — (2)$$

<u>Theorem (Lax-Milgram)</u>

Let B be a bounded, coercive bilinear form on a Hilbert space $\mathcal{H}$. Then for any $F \in \mathcal{H}^*$ there exists a unique $f \in \mathcal{H}$ s.t.

$$B(x,f) = F(x) \quad \text{for all } x \in \mathcal{H}.$$

NB: There is $\underline{\underline{\text{no}}}$ requirement for B to be symmetric.

Now consider an elliptic operator

$$Lu = -D_i(a^{ij} D_j u) + b^i D_i u + cu,$$

where $a^{ij}, b^i, c \in L^\infty(\Omega)$ and a^{ij} are uniformly elliptic with ellipticity constant $\lambda > 0$.

Let the associated bilinear form be given by

$$B(u,v) = \int_\Omega (a^{ij} D_j u\, D_i v + b^i D_i u\, v + c\, uv)\, dx$$

for $u, v \in W_0^{1,2}(\Omega)$.

We note that

$$|B(u,v)| \le \sum_{i,j} \|a^{ij}\|_\infty \int_\Omega |Du||Dv|\,dx + \sum_i \|b^i\|_\infty \int_\Omega |Du||v|\,dx + \|c\|_\infty \int_\Omega |uv|\,dx$$

$$\le K \|u\|_{1,2;\Omega} \|v\|_{1,2;\Omega}$$

and

$$\lambda \int_\Omega |Du|^2\,dx \le \int_\Omega a^{ij} D_j u\, D_i v\,dx$$

$$= B(u,v) - \int_\Omega \Big(\sum_i b^i D_i u\, u + cu^2 \Big)\,dx$$

$$\le B(u,u) + \sum_i \|b^i\|_\infty \int_\Omega |Du||u|\,dx + \|c\|_\infty \int_\Omega |u|^2\,dx$$

$$\le B(u,u) + \hat{K}\varepsilon \int_\Omega |Du|^2\,dx + \Big(\frac{\hat{K}}{4\varepsilon} + \|c\|_\infty \Big) \int_\Omega |u|^2\,dx$$

$$\Rightarrow \frac{\lambda}{2} \int_\Omega |Du|^2\,dx \le B(u,u) + \hat{C} \int_\Omega |u|^2\,dx$$

for some $\hat{C} \ge 0$.

Moreover, by the Poincaré inequality(†) we have

$$\beta \|u\|^2_{1,2;\Omega} \leq B(u,u) + \sigma \|u\|^2_{2;\Omega}$$

for some $\beta > 0$ and some $\sigma \geq 0$.

Theorem. There is some $\sigma \geq 0$ such that for each $\mu \geq \sigma$ and each $f \in L^2(\Omega)$ there exists a weak solution $u \in W^{1,2}_0(\Omega)$ of

$$\mathcal{L}u + \mu u = f \text{ in } \Omega.$$

Proof. Define $B_\mu(u,v) = B(u,v) + \mu \langle u,v \rangle$ (i.e. $\mathcal{L}_\mu u = \mathcal{L}u + \mu u$).

Then B_μ satisfies the assumptions of the Lax-Milgram theorem with

$$\boxed{(\mu - \sigma)\|u\|^2_{2;\Omega} + \beta \|u\|^2_{1,2;\Omega} \leq B_\mu(u,u)}$$

for any $u \in W^{1,2}_0(\Omega)$.

Now any $f \in L^2(\Omega)$ acts on $W^{1,2}_0(\Omega)$ as a bounded linear functional according to

$$F : W^{1,2}_0(\Omega) \ni v \mapsto \int_\Omega f v \, dx. \qquad - (*)$$

Then the Lax-Milgram theorem implies that there exists a unique $u \in W^{1,2}_0(\Omega)$ s.t.

$$B_\mu(u,v) = \langle f, v \rangle \text{ for all } v \in W^{1,2}_0(\Omega). \quad ■$$

(†) i.e. $\|u\|_{2;\Omega} \leq C \|Du\|_{2;\Omega}$ for $u \in W^{1,2}_0(\Omega)$.

Remark. We would like solvability for not just μ sufficiently large but solvability for all but a countable discrete set of μ.

§ Fredholm theory

Let $\mathcal{H} = W^{1,2}_0(\Omega)$ and define the embedding $I : \mathcal{H} \to \mathcal{H}^*$ by

$$Iu : v \mapsto \int_\Omega u v \, dx. \qquad - (**)$$

Then we:

Claim. The mapping I is compact.

Proof. We may write $I = I_1 \circ I_2$ where $I_2 : \mathcal{H} \to L^2(\Omega)$ is the canonical embedding and $I_1 : L^2(\Omega) \to \mathcal{H}^*$ is given by $(**)$.

By the Kondraschov's compactness result, I_2 is compact and as I_1 is clearly continuous, I must be compact. ■

Now choose some $\mu > 0$ so that $\mathcal{L}_\mu u = \mathcal{L}u + \mu u$ is bounded and coercive on the Hilbert space $\mathcal{H}$.

Then for $u \in \mathcal{H}$ and $F \in \mathcal{H}^*$ given by $(*)$, we note that

$$\boxed{\mathcal{L}u = F} \equiv \boxed{\mathcal{L}_\mu u - \mu I u = F}$$

i.e. $B(u,v) = F(v) \iff B_\mu(u,v) = (F + \mu I u)(v)$.

and, moreover, by the above there is some $\ell > 0$ s.t.

$$\ell \|u\|_{1,2}^2 \le B_\mu(u,u) = \langle f, u\rangle \le \|f\|_2 \|u\|_{1,2}$$

$$\Rightarrow \|u\|_{1,2;\Omega} \le C \|f\|_{2;\Omega} \quad \text{for some } c > 0.$$

Thus by the previous theorem, $\mathcal{L}_\mu^{-1} : \mathcal{H}^* \to \mathcal{H}$ is a bounded one-to-one map.

Hence

$$u - \mu \mathcal{L}_\mu^{-1} I u = \mathcal{L}_\mu^{-1} F$$

and $\boxed{T = \mu \mathcal{L}_\mu^{-1} I}$ is compact by the above claim.

Therefore, by the Fredholm alternative :

Theorem. Precisely one of the following statements holds. Either :

(α) | $\forall f \in L^2(\Omega)$, $\exists!$ weak solution $u \in W_0^{1,2}(\Omega)$ of $\mathcal{L}u = f$ in Ω. |

or

(β) | $\exists$ non-trivial weak solutions $u \in W_0^{1,2}(\Omega)$ of $\mathcal{L}u = 0$ in Ω. |

Remark. If the assertion (β) holds, the dimension of the subspace $N \subset W_0^{1,2}(\Omega)$ of weak solutions to $\mathcal{L}u = 0$ in Ω is finite.

§ Uniqueness.

Theorem (Max. principle for weak solutions)

Let $u \in W_0^{1,2}(\Omega)$ satisfy $\quad \mathcal{L}u \le 0 \ (\ge 0)$ in Ω under the additional assumption that $\boxed{c \ge 0.}$

Then

$$\operatorname*{ess\,sup}_{\Omega} u \le \operatorname*{ess\,sup}_{\partial\Omega} u^+ \quad \left(\operatorname*{ess\,inf}_{\Omega} u \ge \inf_{\partial\Omega} u^- \right).$$

Remark. Note that

$$\operatorname*{ess\,sup}_{\partial\Omega} u = \inf\{k : u \le k \text{ on } \partial\Omega, k \in \mathbb{R}\},$$

$$\operatorname*{ess\,inf}_{\partial\Omega} u = -\operatorname*{ess\,sup}_{\partial\Omega}(-u)$$

and we say $u \in W^{1,2}(\Omega)$ satisfies "$u \le 0$ in Ω" if $\quad u^+ = \max(u,0) \in W_0^{1,2}(\Omega)$.

Corollary. If $u \in W_0^{1,2}(\Omega)$ satisfies $\mathcal{L}u = 0$ in Ω with $c \ge 0$, then $u = 0$ in Ω.

[Thus we have the uniqueness of the trivial solution $\mathcal{L}u = 0$ in the case $c \ge 0$ and so solutions to $\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ is uniquely solvable by the Fredholm alternative theorem.]