SPECTRAL THEORY FOR ELLIPTIC OPERATORS

§ Spectral Theory : A review.

Def. Let T be a bounded linear operator on a Banach space X. Then

(i) The resolvent set of T is

$$\rho(T) = \{ \lambda \in \mathbb{C} : (\lambda - T) \text{ is one-to-one \& onto} \}.$$

(ii) The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

(iii) the point spectrum of T is

$$\sigma_p(T) = \{ \lambda \in \mathbb{C} : (\lambda - T) \text{ is } \underline{\text{not}} \text{ one-to-one} \}$$

Theorem (Riesz-Schauder)

If T is a compact operator, then $\sigma(T)$ is at most countable with 0 being the only possible accumulation point. Furthermore, any non-zero point of $\sigma(T)$ is an eigenvalue of finite multiplicity.

Corollary (Fredholm alternative)

If T is a compact operator and $\lambda \neq 0$, then

$$\lambda \in \rho(T) \cup \sigma_p(T).$$

That is to say, either

(i) The homogeneous equation $(\lambda - T)x = 0$ has a non-trivial solution $x \in X$.

OR

(ii) For each $y \in X$ the equation $(\lambda - T)x = y$ has a uniquely determined solution $x \in X$ and $(\lambda - T)^{-1}$ is bounded.

By denoting

$$N(T) = \{ x \in X : Tx = 0 \} \quad (\text{kernel})$$

$$R(T) = \{ y \in X : y = Tx \text{ for some } x \in X \} \quad (\text{range})$$

we also have:

Theorem (Fredholm operators)

Let T be a compact operator and $\lambda \in \mathbb{C} \setminus \{0\}$.

Then

(i) $N(\lambda - T)$ is finite dimensional.

(ii) $R(\lambda - T)$ is closed of finite codimension.

(iii) $\dim N(\lambda - T) = \dim N(\lambda - T^*) = \operatorname{codim}(\lambda - T) = \operatorname{codim}(\lambda - T^*)$.

Remark. In the Hilbert space case we also have in general

$\overline{R(A)} = N(A)^{\perp}$ for any bounded linear $A : \mathcal{H} \to \mathcal{H}$.

§ Existence revisited.

Let $\mathcal{L} = -\partial_i(a^{ij}\partial_j u) + b^i\partial_i u + cu$, where $a^{ij}, b^i, c \in L^\infty(\Omega)$ and a^{ij} are uniformly elliptic.

Theorem (Existence, 3rd version)

There exists at most a countable discrete set $\Sigma \subset \mathbb{R}$ such that if $\lambda \notin \Sigma$, the boundary value problem

$$\begin{cases} \mathcal{L}u = \lambda u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad - (*)$$

is uniquely solvable for arbitrary $f \in L^2(\Omega)$.

If $\lambda \in \Sigma$, the subspace of solutions to the homogeneous problem

$$\begin{cases} \mathcal{L}u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is of a non-zero finite dimension.

Proof. By the previous lecture we have

$$(\mu - \sigma)\|u\|_2^2 + \beta \|u\|_{1,2}^2 \le B_\mu(u,u)$$

for each $\mu \ge \sigma$, some $\beta > 0$ and (w.l.o.g) some $\sigma > 0$.

[i.e. take $\boxed{\mu = \sigma > 0}$ so that $\beta\|u\|_{1,2}^2 \le B_\sigma(u,u)$]

Case 1. (Assume $\lambda \neq -\sigma$).

By the Fredholm alternative, $(*)$ has a unique weak solution for each $f \in L^2(\Omega)$ iff $u=0$ is the only solution of

$$\begin{cases} \mathcal{L}u - \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

This is true iff $u=0$ is the only weak solution of

$$\begin{cases} \mathcal{L}_\sigma u = (\sigma+\lambda) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad - (**)$$

where $\mathcal{L}_\sigma = \mathcal{L}u + \sigma u$.

The latter holds exactly when $Tu = \frac{\sigma}{\sigma+\lambda} u$, (NB: $\sigma+\lambda \neq 0$ by assumption)

where as before $T = \sigma \mathcal{L}_\sigma^{-1} I$ is a compact operator on $L^2(\Omega)$.

Thus $\frac{\sigma}{\sigma+\lambda} \in \rho(T)$ is not an eigenvalue of T.

So $(*)$ has a unique solution for each $f \in L^2(\Omega)$ iff $\lambda \notin \Sigma = \{\sigma(\frac{1}{\kappa} - 1) : \kappa \in \sigma(T)\}$.

Case 2. (Assume $\lambda = -\sigma$)

We note that $(**)$ becomes now

$$\begin{cases} \mathcal{L}_\sigma u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

i.e. $u = T(0)$ which by linearity implies $u = 0$.

Moreover $-\sigma \notin \Sigma$ as $\sigma(T)$ is a bounded subset. ∎

§ Self-adjoint operators

Let $T: \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint (i.e. $T = T^*$) linear operator on a Hilbert space $\mathcal{H}$ and let

$$m = \inf_{\substack{u \in \mathcal{H} \\ \|u\| = 1}} \langle Tu, u \rangle, \qquad M = \sup_{\substack{u \in \mathcal{H} \\ \|u\| = 1}} \langle Tu, u \rangle$$

Then we have:

Lemma. If $T: \mathcal{H} \to \mathcal{H}$ is a bounded self-adjoint linear operator, then

(1) $\sigma(T) \subset \mathbb{R}$

(2) $m, M \in \sigma(T)$

(i.e. the spectrum is real and bounded)

Theorem (Hilbert-Schmidt)

Let T be a self-adjoint compact operator on a separable Hilbert space $\mathcal{H}$. Then there exists an o/n basis (e_j) of $\mathcal{H}$ s.t. $Te_j = \lambda_j e_j$ where $\lambda_j \to 0$.

(NB: $\lambda_j \to 0$ follows from the Riesz-Schauder theorem)

§ Spectral theory for symmetric elliptic operators

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and

$$\mathcal{L} = - D_i (a^{ij} D_j u)$$

be a symmetric elliptic operator (i.e. $a^{ij} = a^{ji}$ & a^{ij} uniformly elliptic).

We have shown that there is a solution operator $S: L^2(\Omega) \to W_0^{1,2}(\Omega)$ to the problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (\text{i.e. } Sf = u).$$

By the Kondrachov compactness theorem, we can consider $S = \mathcal{L}^{-1}$ as a compact operator $L^2(\Omega) \to L^2(\Omega)$.

Under the above assumptions, the operator S is self-adjoint (in the sense that $\langle Su, v \rangle = \langle u, Sv \rangle$, $\forall u, v \in L^2(\Omega)$)

Then by the Hilbert-Schmidt theorem, we have:

Theorem. (1) Each eigenvalue of $\mathcal{L}$ is real.

(2) $\Sigma = \{\lambda_j\}_{j \geq 1}^{\infty}$ can be ordered so that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

where $\lambda_j \to \infty$ as $j \to \infty$

(3) There exists o/n basis $\{u_j\}_{j \in \mathbb{N}}$ of $L^2(\Omega)$, where $u_j \in W_0^{1,2}(\Omega)$, for each $j \geq 1$, is an eigenfunction corresponding to eigenvalue λ_j that solves

$$\begin{cases} \mathcal{L}u_j = \lambda_j u_j & \text{in } \Omega \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

NB: For $\mu \neq 0$ we have $Su = \mu u \iff \mathcal{L}u = \lambda u$ for $\lambda = 1/\mu$.

§ Weyl's law

Q: What is the asymptotic behaviour of the eigenvalues of the Laplacian operator Δ in a bounded domain $\Omega \subset \mathbb{R}^n$ with "sufficiently smooth" boundary?

For the Dirichlet boundary case

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{— } (*)$$

we have a discrete spectrum of positive eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

The asymptotic behaviour of the eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ can be deduced from the "counting function"

$$N(\lambda) = \text{no. of positive eigenvalues} \le \lambda \quad (\text{counted with multiplicity})$$

$$= \sum_{\lambda_j \le \lambda} 1 . \qquad (\text{NB: } N(\lambda_j) = j)$$

Weyl's conjecture:

$$\boxed{N(\lambda) = \frac{\omega_n}{(2\pi)^n} |\Omega| \lambda^{n/2} - \frac{1}{4} \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\partial\Omega| \lambda^{\frac{n-1}{2}} + o(\lambda^{\frac{n-1}{2}})}$$

where $\omega_n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ = volume of unit n-ball.

§ Toy model: The rectangular box.

Theorem. For the problem $(*)$ on a square domain $\Omega = [0,a]^2 \subset \mathbb{R}^2$, we have

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{4\pi} .$$

Proof. By the method of separation of variables, we can explicitly compute the eigenfunctions & eigenvalues of the problem $(*)$ when $\Omega = [0,a]^2$.

The eigenfunctions are given by

$$\sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{\ell \pi y}{a}\right), \qquad k = 1, 2, \ldots \quad \ell = 1, 2, \ldots$$

with the corresponding eigenvalues

$$\frac{\pi^2}{a^2}(k^2 + \ell^2).$$

Hence $N(\lambda)$ is equal to the number of positive integer lattice points $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ s.t. $k^2 + \ell^2 \le \lambda \frac{a^2}{\pi^2}$.
This is equivalent to counting $\ell = 1, 2, \ldots, \lfloor \sqrt{\Lambda^2 - k^2} \rfloor$ for each $k = 1, 2, \ldots, \lfloor \Lambda \rfloor$, where $\Lambda = \frac{a}{\pi}\sqrt{\lambda}$. (†)
Thus we have

$$N(\lambda) = \sum_{k=1}^{\lfloor \Lambda \rfloor} \lfloor \sqrt{\Lambda^2 - k^2} \rfloor = \sum_{k=1}^{\lfloor \Lambda \rfloor} \sqrt{\Lambda^2 - k^2} + O(\Lambda)$$

$$= \Lambda^2 \left(\frac{1}{\Lambda} \sum_{k=1}^{\lfloor \Lambda \rfloor} \sqrt{1 - \frac{k^2}{\Lambda^2}} \right) + O(\Lambda)$$

$$= \Lambda^2 \left(\int_0^1 \sqrt{1 - t^2}\, dt + o(1) \right) + O(\Lambda)$$

$$= \frac{\pi}{4} \Lambda^2 + O(\Lambda).$$

Therefore $\boxed{N(\lambda) = \frac{a^2}{4\pi}\lambda + O(\sqrt{\lambda})}$. ∎

(†) Here $\lfloor x \rfloor$ = floor(x).