SPECTRAL THEORY : EXAMPLE

§ Vibrating Membrane : General Case

The PDE for a vibrating homogeneous membrane in 2-dimensions with a clamped boundary condition is given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u & t \geq 0, \quad x \in \Omega \subset \mathbb{R}^2 \\ u = 0 & x \in \partial\Omega \end{cases}$$

If we set $u(x,y,t) = v(x,y)\, g(t)$, then it follows that

$$\frac{\Delta v}{v} = \frac{\ddot{g}}{g} = -\lambda$$

from which it follows that λ must be a constant.

Thus we want to solve the eigenvalue problem

$$\begin{cases} \Delta v + \lambda v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

The eigenvalue λ must in fact be positive, since

$$\int_\Omega (v_x^2 + v_y^2)\, dx\,dy = -\int_\Omega v\, \Delta v \, dx\,dy = \lambda \int_\Omega v^2 \, dx\,dy .$$

and the general solution of $\ddot{g}/g = -\lambda$ has the form

$$g = a \cos(\sqrt{\lambda}\, t) + b \sin(\sqrt{\lambda}\, t) .$$

Then the solution

$$u(x,y,t) = v(x,y)\left(a \cos(\sqrt{\lambda}\, t) + b \sin(\sqrt{\lambda}\, t)\right)$$

corresponds to an eigenvibration of frequency $\sqrt{\lambda}$.

From the general theory, there is a countably infinite sequence of eigenvalues $\lambda_1, \lambda_2, \ldots$ with corresponding eigenvalues $v_1, v_2, \ldots$.

NB: Eigenfunctions v_j, v_k corresponding to eigenvalues λ_j and λ_k are orthogonal, since

$$(\lambda_j - \lambda_k) \int_\Omega v_j v_k \, dx\,dy = -\int_\Omega (v_k \Delta v_j - v_j \Delta v_k)\, dx\,dy = 0$$

§ General Solution.

The motion of a freely vibrating membrane with arbitrary initial conditions:

$$\begin{cases} u(x,y,0) = f(x,y) \\ \dfrac{\partial u}{\partial t}(x,y,0) = g(x,y) \end{cases}$$

can be represented by the eigenfunction expansion

$$u(x,y,t) = \sum_{n=1}^{\infty} v_n(x,y) \left[a_n \cos(\sqrt{\lambda_n}\, t) + b_n \sin(\sqrt{\lambda_n}\, t) \right]$$

where

$$a_n = \int_\Omega f\, v_n \, dx\,dy, \quad b_n = \frac{1}{\sqrt{\lambda_n}} \int_\Omega g\, v_n \, dx\,dy$$

Here $\sqrt{\lambda_n}$ are the frequencies of the corresponding eigenvibrations.

Remark (Nodal lines).

The points at which an eigenfunction $v_n(x,y)=0$ are called "nodal lines". They are the curves along which the membrane remains at rest during the eigenvibration.

§ Circular membrane problem and Bessel functions

Let now Ω = 2-dimensional unit disk.

The PDE for the eigenvalue problem (in polar coord.) takes the form

$$\begin{cases} v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} + \lambda v = 0 \\ v(1,\theta) = 0 \end{cases}$$

By setting $v(r,\theta) = f(r) h(\theta)$ we find

$$\frac{r^2\left(f''(r) + \frac{1}{r} f'(r) + \lambda f(r)\right)}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = \text{const.}$$

As $h(\theta)$ must be a periodic function (to ensure v is single valued) we must have const. $= n^2$ $(n=0,1,2,\ldots)$.

Hence $$h(\theta) = a\cos n\theta + b\sin n\theta,$$

and by setting $f(r)=y$, we get the following

$$r^2 y'' + r y' + (r^2\lambda - n^2) y = 0. \qquad (\dagger)$$

The problem is to find eigenvalues λ for which there exists solutions to $(\dagger)$ that also satisfy the boundary condition $y=0$.

By making the transform $r\sqrt{\lambda} = z$, $\lambda \neq 0$, the equation $(\dagger)$ takes the form

$$z^2 y''(z) + z y'(z) + (z^2 - n^2) y(z) = 0. \quad \text{(Bessel's eqn)}$$

Bounded solution are given by the Bessel functions of the 1st kind $J_{\pm n}(z)$, i.e. solutions of the form

$$y(z) = J_n(z) = \left(\tfrac{1}{2}z\right)^n \sum_{l=0}^{\infty} \frac{\left(-\frac{1}{4}z^2\right)^l}{l!\,\Gamma(n+l+1)}$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - n\theta)\,d\theta.$$

In which case solutions of $(\dagger)$ can be written in the form

$$y_n(r) = J_n(\sqrt{\lambda}\, r)$$

where λ is determined by the boundary condition $y_n(1)=0$, i.e. the zeros of $J_n(\cdot)$.

Hence the eigenvalues λ are the square of the zeros of the Bessel functions J_n.

Now $J_n(z)$ has real zeros for $n \in \mathbb{R}$; all singular except when $z=0$.

For $n \geq 0$ the positive zeros of $J_n(z)$ are denoted by

$$\boxed{j_{n,s}} \qquad (s = 1, 2, 3 \ldots)$$

we can in fact write

$$J_n(z) = \frac{(\frac{1}{2}z)^n}{\Gamma(n+1)} \prod_{s=1}^{\infty} \left[1 - \left(\frac{z}{j_{n,s}}\right)^2\right].$$

The eigenfunctions v_n can then be written in the form

$$v_n(r, \theta) = J_n(j_{n,s} r)(\alpha \cos n\theta + \beta \sin n\theta)$$

- The constants α, β are still arbitrary, indicating the eigenvalues are double, since they have linearly independent eigenfunctions $J_n \cos n\theta$, $J_n \sin n\theta$
- Only when v_n is rotationally symmetric are the eigenvalues singular.
- The first few eigenvalues λ are

$$j_{0,1}^2 \quad \underbrace{j_{1,1}^2 \quad j_{1,1}^2} \quad \underbrace{j_{2,1}^2 \quad j_{2,1}^2} \quad j_{0,2}^2 \quad \underbrace{j_{3,1}^2 \quad j_{3,1}^2} \quad \underbrace{j_{1,2}^2 \quad j_{1,2}^2} \quad \text{etc} \ldots$$

- The nodal lines for the eigenfunctions are circles $\rho = \text{const.}$ and radial lines $\theta = \text{const.}$

The eigenvibrations are represented by

$$u = J_n(j_{n,s} r)(\alpha \cos n\theta + \beta \sin n\theta)(a \cos(j_{n,s} t) + b \sin(j_{n,s} t))$$

The frequencies of the lowest eigenvibrations are:

$$j_{0,1} = 2.404$$

$$j_{1,1} = 3.832$$

$$j_{2,1} = 5.135$$

$$j_{0,2} = 5.52$$

$$j_{3,1} = 6.38$$

$$j_{1,2} = 7.016$$

$$j_{4,1} = 7.588$$

$$\vdots$$

The frequencies corresponding to the fundamental eigenvibrations do not belong to a harmonic scale, however there are some close approximations:

$$\begin{cases} \frac{4}{3}\frac{j_{1,1}}{j_{0,1}} = 2.124 \simeq \frac{j_{2,1}}{j_{0,1}} = 2.136 \\[2ex] \frac{5}{3}\frac{j_{1,1}}{j_{0,1}} = 2.656 \simeq \frac{j_{3,1}}{j_{0,1}} = 2.653 \\[2ex] 2\frac{j_{1,1}}{j_{0,1}} = 3.187 \simeq \frac{j_{4,1}}{j_{0,1}} = 3.155 \end{cases}$$

where $\frac{j_{1,1}}{j_{0,1}} = 1.593$

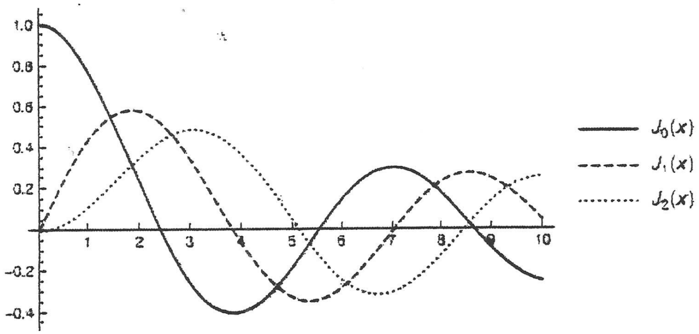
Table 9.5

ZEROS AND ASSOCIATED VALUES OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

s	$j_{0,s}$	$j_{1,s}$	$j_{2,s}$
1	2.40482 55577	3.83171	5.13562
2	5.52007 81103	7.01559	8.41724
3	8.65372 79129	10.17347	11.61984
4	11.79153 44391	13.32369	14.79595
5	14.93091 77086	16.47063	17.95982
6	18.07106 39679	19.61586	21.11700
7	21.21163 66299	22.76008	24.27011
8	24.35247 15308	25.90367	27.42057
9	27.49347 91320	29.04683	30.56920
10	30.63460 64684	32.18968	33.71652
11	33.77582 02136	35.33231	36.86286
12	36.91709 83537	38.47477	40.00845
13	40.05842 57646	41.61709	43.15345
14	43.19979 17132	44.75932	46.29800
15	46.34118 83717	47.90146	49.44216
16	49.48260 98974	51.04354	52.58602
17	52.62405 18411	54.18555	55.72963
18	55.76551 07550	57.32753	58.87302
19	58.90698 39261	60.46946	62.01622
20	62.04846 91902	63.61136	65.15927

s	$j_{3,s}$	$j_{4,s}$	$j_{5,s}$
1	6.38016	7.58834	8.77148
2	9.76102	11.06471	12.33860
3	13.01520	14.37254	15.70017
4	16.22347	17.61597	18.98013
5	19.40942	20.82693	22.21780
6	22.58273	24.01902	25.43034
7	25.74817	27.19909	28.62662
8	28.90835	30.37101	31.81172
9	32.06485	33.53714	34.98878
10	35.21867	36.69900	38.15987
11	38.37047	39.85763	41.32638
12	41.52072	43.01374	44.48932
13	44.66974	46.16785	47.64940
14	47.81779	49.32036	50.80717
15	50.96503	52.47155	53.96303
16	54.11162	55.62165	57.11730
17	57.25765	58.77084	60.27025
18	60.40322	61.91925	63.42205
19	63.54840	65.06700	66.57289
20	66.69324	68.21417	69.72289

s	$j_{6,s}$	$j_{7,s}$	$j_{8,s}$
1	9.93611	11.08637	12.22509
2	13.58929	14.82127	16.03777
3	17.00382	18.28758	19.55454
4	20.32079	21.64154	22.94517
5	23.58608	24.93493	26.26681
6	26.82015	28.19119	29.54566
7	30.03372	31.42279	32.79580
8	33.23304	34.63709	36.02562
9	36.42202	37.83872	39.24045
10	39.60324	41.03077	42.44389
11	42.77848	44.21541	45.63844
12	45.94902	47.39417	48.82593
13	49.11577	50.56818	52.00769
14	52.27945	53.73833	55.18475
15	55.44059	56.90525	58.35789
16	58.59961	60.06948	61.52774
17	61.75682	63.23142	64.69478
18	64.91251	66.39141	67.85943
19	68.06689	69.54971	71.02200
20	71.22013	72.70655	74.18277



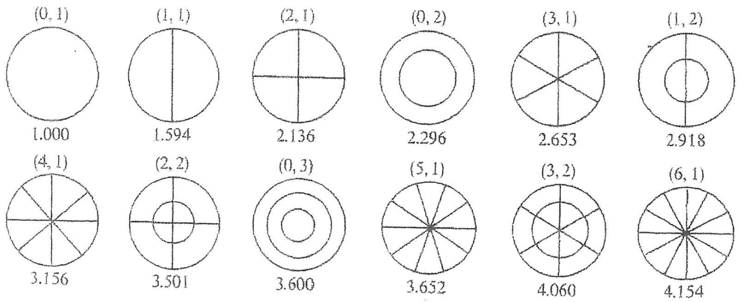


Fig. 3.6. First 12 modes of an ideal membrane. The mode designation (m, n) is given above each figure and the relative frequency below. To convert these to actual fre-

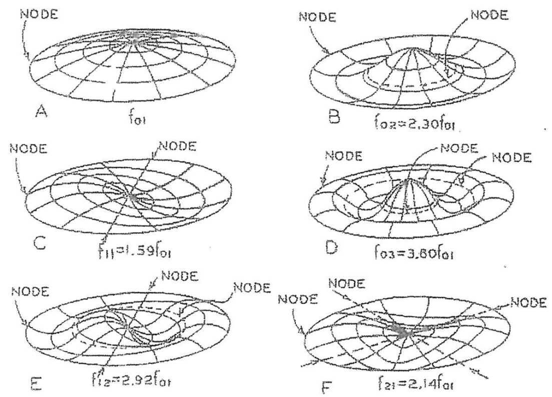


Fig. 4.8. Modes of vibration of a stretched circular membrane. (*After Morse Vibration and Sound, McGraw-Hill Book Company, Inc., New York, 1948.*)