# Approximation Methods for Quadratic Bézier Curve, by Circular Arcs within a Tolerance Band

*Seminar aus Informatik*
Univ.-Prof. Dr. Wolfgang Pree
Seyed Amir Hossein Siahposhha

**Keyword:**

Quadratic Bézier Curve, Bisection Algorithm, Biarc, Biarc Method, Hausdorff Distances, Tolerance Band.

**Introduction:**

General curves can be created by any type of algebraic formula but in engineering and industry the polynomial and rational curves are common to use because they are simpler and easier to program.

The piecewise Bézier curve is a kind of general curve (polynomial) in Computational Geometry, which is widely used in various interesting CAD/CAM and Computer Graphics applications (Figure 1.a).

On the other hand, in industry, the tool path for CNC (Computer Numerical Control) machinery because of physical limitation is able to use just the piecewise circular curve and straight line segments (Figure 1.b). There are several methods to approximate Bézier Curve by line segments, but designed shapes consisting of tangential circular arc segments are better and easier to use than the polygons consisting of line segments as the tool path of CNC machinery. The combination of these two topics is the main reason of over twenty years attempting and research on approximation of Bézier curve by tangential circular arcs.

In programming the tool path of CNC machinery, a smaller number of arc segments in approximation can reduce the number of instructions and tool motions. Therefore, to improve production efficiency we need to approximate Bézier curves by tangential circular arc segments with fewer arc segments that are as small as possible.

Circular arc can be exactly represented by using standard (polynomial) or rational cubic Bézier curve and rational quadratic Bézier curves but not with standard quadratic Bézier curves. Every Bézier curve can be just represented as a Bézier curve of higher degree. Therefore we cannot use the cubic or high degrees Bézier curves approximation methods for quadratic Bézier curve and that makes the main challenge of approximation for quadratic Bézier curve.
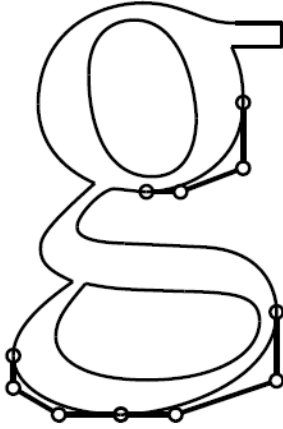
Figure 1:    a) Designing by Bézier curve.  b) CNC machining tool path.

## 1.   Bézier Curve

The pioneers of Bézier Curve were Pierre Bézier who worked as an engineer at Renault, and Paul de-Casteljau who worked at Citroen; this reflects the usability and importance of these classic curves in industry and engineering.
General curves can be created by any type of algebraic formula but applied mathematics is not able to do every processing on every formula. Because of this huge limitation, just the simple curves that is easy to process are using in engineering and industry.

### 1.1.   Bézier Curve Formulation

Bézier curve can represent in different degrees and there are several methods to formulate Bézier curve, for example:

  I.   Linear Bézier Curve
$$\mathbf{B}(t) = \mathbf{P_0} + t(\mathbf{P_1} - \mathbf{P_0}) = (1-t)\mathbf{P_0} + t\mathbf{P_1} \ , \ t \in [0,1]$$

  II.   Quadratic Bézier Curve
$$\mathbf{B}(t) = (1-t)^2\mathbf{P_0} + 2(1-t)t\mathbf{P_1} + t^2\mathbf{P_2} \ , \ t \in [0,1].$$
$$\mathbf{B}(t) = (1-t)^3\mathbf{P_0} + 3(1-t)^2t\mathbf{P_1} + 3(1-t)t^2\mathbf{P_2} + t^3\mathbf{P_3} \ , \ t \in [0,1].$$

  III.   Cubic Bézier Curves
$$\mathbf{B}(t) = (1-t)[(1-t)\mathbf{P_0} + t\mathbf{P_1}] + t[(1-t)\mathbf{P_1} + t\mathbf{P_2}] \ , \ t \in [0,1]$$
$$\mathbf{B}(t) = (1-t)\mathbf{B_{P_0,P_1,P_2}}(t) + t\mathbf{B_{P_1,P_2,P_3}}(t) \ , \ t \in [0,1].$$

  IV.   Quartic Bézier Curve
$$\mathbf{B}(t) = (1-t)\mathbf{B_{P_0,P_1,P_2,P_3}}(t) + t\mathbf{B_{P_1,P_2,P_3,P_4}}(t) \ , \ t \in [0,1].$$

  V.   Fifth-order Bézier Curve
$$\mathbf{B_{P_0P_1P_2P_3P_4P_5}}(t) = \mathbf{B}(t) = (1-t)^5\mathbf{P_0} + 5t(1-t)^4\mathbf{P_1} + 10t^2(1-t)^3\mathbf{P_2}$$
$$+ 10t^3(1-t)^2\mathbf{P_3} + 5t^4(1-t)\mathbf{P_4} + t^5\mathbf{P_5}, \quad t \in [0,1].$$

High degrees Bézier curves are too complex to compute and they are not common to use. The most applicable Bézier curves are Quadratic and Cubic.

Bézier curve can representing recursive by using lower degrees Bézier curves and generate by this general recursive formula.

$$\mathbf{B}(t) = \mathbf{B}_{\mathbf{P_0P_1...P_n}}(t) = (1-t)\mathbf{B}_{\mathbf{P_0P_1...P_{n-1}}}(t) + t\mathbf{B}_{\mathbf{P_1P_2...P_n}}(t)$$

The general recursive formula is extendable to a general formula that called Bernstein Polynomials:

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i}(1-t)^{n-i}t^i\mathbf{P}_i$$

$$= (1-t)^n\mathbf{P}_0 + \binom{n}{1}(1-t)^{n-1}t\mathbf{P}_1 + \cdots$$

$$\cdots + \binom{n}{n-1}(1-t)t^{n-1}\mathbf{P}_{n-1} + t^n\mathbf{P}_n, \quad t \in [0,1],$$

### 1.2.   Bézier Curve and Circular Arc

The high degree Bézier Curves are too complex to process and approximation; therefore the Quadratic and Cubic Bézier curve are more common to use in CAD/CAM.

Circular arc can be exactly represented by Cubic Bézier curve as it is shown in Figure 2.a but it cannot represented by any Quadratic Bézier curve because The Quadratic Bézier curves is a parabolic segment which is one of the those conic section (Figure 2.b). Therefore there are more different methods to approximate the Cubic Bézier curve.
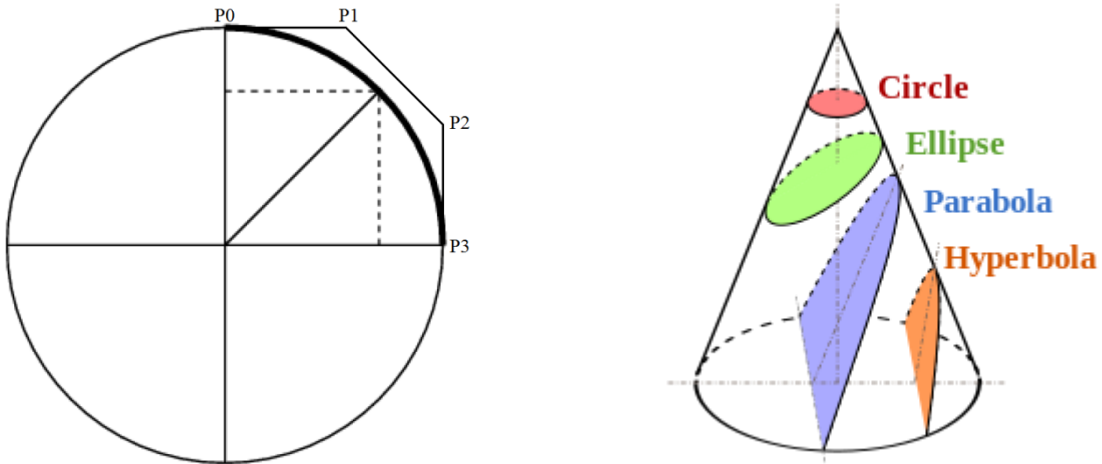


Figure 2: a) Cubic Bézier curve can represent exact circular arc. b) Conic Sections.

### 1.3. Bézier Curve Construction Algorithm

Every Bézier Curve is dividable to Bézier sub-curves with the same degree by intersection of tangents lines (Figure 3). This fractal features is the base of Bézier Curve Construction Algorithm.
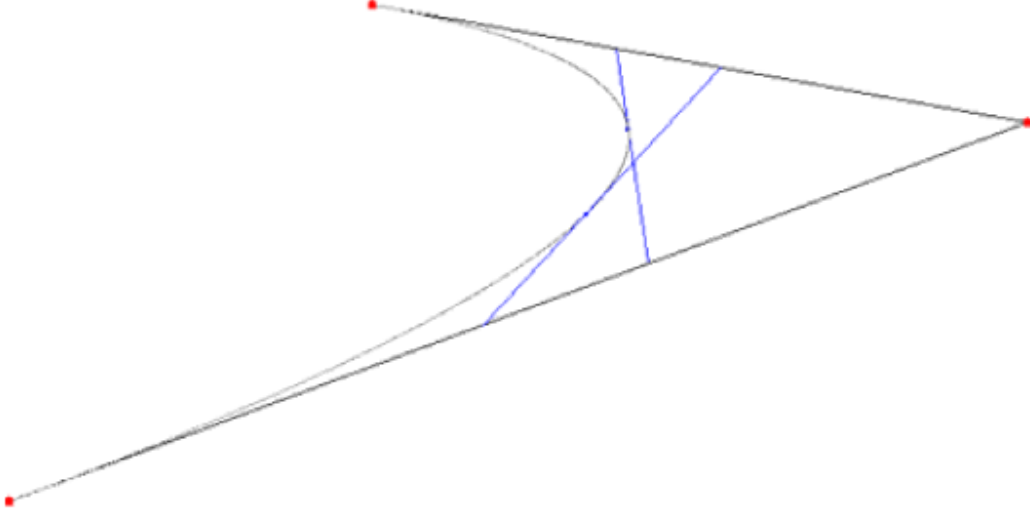


Figure 3: Subdividing Bézier curves by tangents lines (fractal features)

### 1.4. The Bisection (de-Casteljau) Algorithm

The most fundamental algorithm for dealing with Bézier curves is the Bisection algorithm. This was devised by Paul de-Casteljau and is referred to as the de-Casteljau algorithm, which is known as the fundamental geometric construction algorithm. Almost methods for approximating Bézier curves are using the de-Castlejau algorithm as described by the pseudo code.

Consider a Bézier curve defined over the parameter interval$[0, 1]$. It is possible to subdivide such a curve into two new Bézier curves, one of them over the domain $0 \leq t \leq \tau$ and the other over $\tau \leq t \leq 1$.

**Quadratic Bézier Curve ($P_0^0, P_0^1, P_0^2$)**

{

$P_1^0 \leftarrow \dfrac{P_0^0 - P_0^1}{2} + P_0^1;$

$P_1^1 \leftarrow \dfrac{P_0^1 - P_0^2}{2} + P_0^2;$

$P_2^0 \leftarrow \dfrac{P_1^0 - P_1^1}{2} + P_1^1;$

$Draw(P_2^0);$

**Quadratic Bézier Curve ($P_0^0, P_1^0, P_2^0$)**

**Quadratic Bézier Curve ($P_2^0, P_1^1, P_0^2$)**

}

These two new Bézier curves, considered together, are equivalent to the single original curve from which they were derived.

It can be shown that if a curve is repeatedly subdivided, the resulting collection of control points converges to the curve.

Thus, one way of plotting a Bézier curve is to simply subdivide it an appropriate number of times and plot the control polygons.
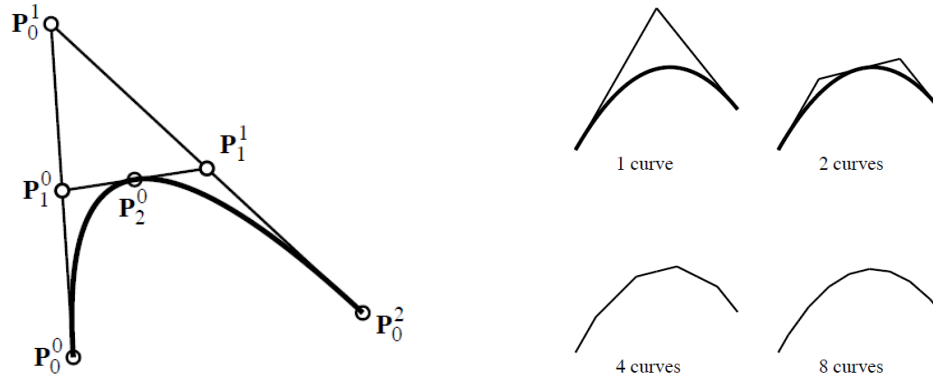


Figure 4: Recursively subdividing a quadratic Bézier curve

Figure 4 shows a quadratic Bézier curve "subdivided" at the first iteration and when a Bézier curve is repeatedly subdivided, the collections of control polygons converge to the curve.

The de-Casteljau algorithm works even for parameter values outside of the original parameter interval because of one immediate value of the Bézier curve, that is each divided Bézier curve segments by every tangent lines (not necessary to follow the de-Casteljau algorithm) are themselves a Bézier curve by the new control points (intersection points of the tangent lines). This important properties is so useful in approximation of Bézier curve and others algorithms.

## 2. Biarc Method

Biarc method is the most important method to approximate the quadratic Bézier curves by circular arcs , that presented by Walton and Meek [1]; A biarc made of two circular arcs, is associated with each quadratic Bézier curve segment to match both end points and the corresponding tangent vectors simultaneously. Equations 1-3 and Figure 5 describe to find a biarc for quadratic Bézier curve.

Equation 1:
$$a = ||P_0 - P_1||, \quad b = ||P_1 - P_2||, \quad c = ||P_0 - P_2||, \quad \cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$

Equation 2:
$$(\omega a + \omega b)^2 = ((1-\omega)a)^2 + ((1-\omega)b)^2 - 2(1-\omega)^2 ab\cos\theta, 0 \leq \omega \leq 1$$

Equation 3: $\omega = \frac{\pm c(a+b) - c^2}{(a+b)^2 - c^2}, 0 \leq \omega \leq 1$
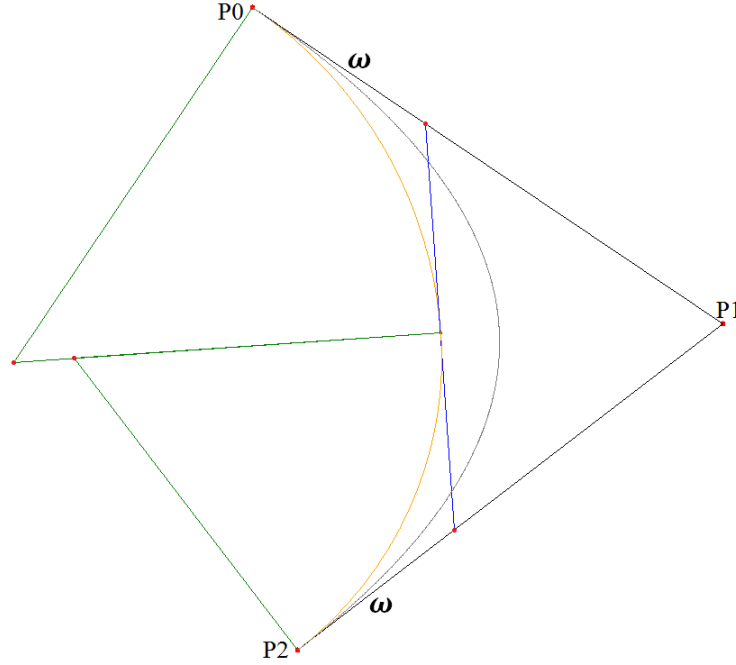
Figure 5: Biarc method

Walton and Meek presented a similar biarc method in [8] to approximate the cubic Bézier curves by $G^1$ Arc-splines.

## 2.1. Hausdorff distance and quadratic Bézier curve approximation

The Hausdorff distance is using in CAD/CAM or Approximation Theories and it is one of the most important parameters for approximating quadratic Bézier curves within a tolerance band.

The Hausdorff distance has obtained from the maximum distance between two curves and it is not easy to find the Hausdorff distance between planar curve and quadratic Bézier curve.

The general mathematic definition for Hausdorff distance is:

Equation 4:

$$d_H(p,q) = \max\left\{\max_{s\in[a,b]}\ \min_{t\in[c,d]}|p(s)-q(t)|, \max_{t\in[c,d]}\ \min_{s\in[a,b]}|p(s)-q(t)|\right\}$$

For more knowledge about the Hausdorff distance and the error measurement between the target curve and approximation curve, refer to [5, 6].

Walton and Meek has presented this formula with $O(n)$ processing time to find the Hausdorff distances of every quadratic Bézier curve segment $Q(u)$ from its biarc $B(C_l, R_l, C_r, R_r)$ measured along a radial direction of the biarc, where $C_l$ and $R_l$ are the centre and radius of the left arc, and $C_r$ and $R_l$ are the centre and radius of the right arc as follow:

Equation 5: $d_H(Q,B) = max_{0\leq u\leq 1}\{\rho_l(u), \rho_r(u)\}$

Equation 6: $\rho_l(u) = \left|R_l - \left\|Q(u) - C_l\right\|\right|, \ 0 \leq u \leq 1$

Equation 7: $\rho_r(u) = \left| R_r - \left\| Q(u) - C_r \right\| \right|, \ 0 \le u \le 1$

Ahn et al. [2] presented a different method that does not follow biarc methods and match one circular arc with each quadratic Bézier curve segment $b_i^N(t)$ recursively, except for the last segment.

Ahn et al.'s method [2] produces the $G^1$ arc spline with fewer arcs than the biarc method [1].

There exists a unique circular arc passing through $\mathbf{a}$ and $\mathbf{b}$ with the unit tangent vector $\mathbf{t_a}$, at $\mathbf{a}$, and the counterclockwise angle from the tangent vector of the circular arc at $\boldsymbol{b}$ to the vector $\boldsymbol{b}$ - $\mathbf{a}$ equals $\boldsymbol{\theta}$ as shown in Figure 6.
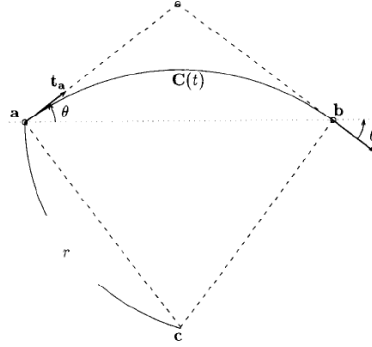


Figure 6: The circular arc is determined uniquely by the two points $\mathbf{a, b}$ and the unit tangent $t_a$ at $\mathbf{a}$

The first arc $C_0^N(t)$ interpolates both end points of the first quadratic Bézier curve segment $b_0^N(t)$ and matches the unit tangent vector of the $b(t)$ at $b_0 = b_0^N$ by such a way as described in Figure 7.a.

The next subsequent arcs, $C_i^N(t), i = 1, ..., N - 1$ interpolate both end points of the quadratic Bézier curve segment $b_i^N(t)$ and match the unit tangent vector $t_i$ of $C_{i-1}^N(t)$ at $C_{i-1}^N(1) = b_{2i}^N$ as shown in Figure 7.a.

The last quadratic Bézier curve segment $b_N^N(t)$ should be approximated by the biarc, which interpolates both end points $b_N^N(t)$ and matches the unit tangent vectors of $C_{N-1}^N(t)$ at $C_{N-1}^N(1) = b_{2N}^N$ and $b_N^N(t)$ at $b_N^N(1) = b_{N+2}^N$ as shown in Figure 7.b, the same as Walton and Meek method.
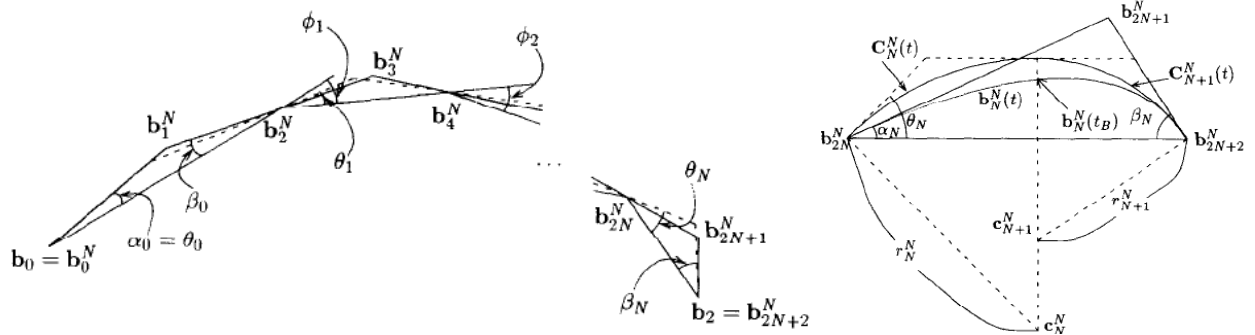


Figure 7: a) $\theta_{i+1} = \phi_{i+1} - \theta_i$ and $\theta_i > 0$ for $i = 0, ..., N - 1$, b) Approximation of the last segment using biarc

They presented a similar method to find the Hausdorff distances in approximating every quadratic Bézier curve segment $Q(u)$ by the matched circular arc $A(C, R)$ as follow:

Equation 8: $d_{Hi}(Q_i, A_i) = \max_{0 \leq u \leq 1}\{\rho_i(u)\}$

Equation 9: $\rho_i(u) = \left|R_i - ||Q_i(u) - C_i||\right|, \ 0 \leq u \leq 1$

Yong et al. [3] introduced two kinds of bisection algorithms that are somehow the generalization and optimizing two previous methods. They have presented the formula of the upper bound for arc segments in the approximate arc spline by Ahn et al.'s algorithm. Based on these formulae, the first bisection algorithm applies the iterative bisection algorithm, searching numbers of the arc segments ranging from two to the upper bound, to get the same arc spline created by Ahn et al.'s algorithm. The other kind of bisection algorithm is based on the idea that to divide the Bézier curve with similar error tolerances between the given curve and the corresponding arcs may reduce arc segments. The curve can be divided in this way by repeatedly bisecting its parametric intervals. It is also proved that the second kind of bisection algorithm can produce the approximate $G^1$ arc spline.
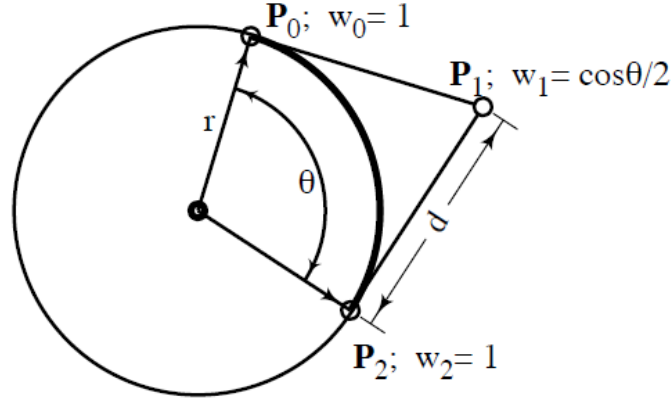


Figure 8: Representing of exact circular arc by quadratic rational Bézier curves

Riskus in [4] has presented a very different method, by using an important property of cubic Bézier curve, which is circular arc can be exactly represented using cubic Bézier curve. There is no such a property like this for standard quadratic Bézier curve and that is possible just for some type of quadratic rational Bézier curves as shown in Figure 8.

Therfore, the Riskus method should be extendible just for this limited type of quadratic rational Bézier curves.

## 2.2.   The Biarc Method Challenges

The first problem of the biarc method is the calculation of Hausdorff Distance. Walton and Meek has presented the formula with $O(n)$ processing time to find the Hausdorff Distances of every quadratic Bézier curve segment from corresponding

biarc. The processing time of this calculation is depending on the program accuracy and it can be a huge amount for the high accuracy.

In additional, the relationship between Hausdorff Distances and Tolerance Band (Figure 9) is an important parameter to determine circular arc number.

In the Biarc method, the Hausdorff Distances can be smaller than or equal to Tolerance Band ($Hd \leq T$). For the minimal number of approximated circular arcs, the Hausdorff Distances must be just equal to Tolerance Band ($Hd = T$).

Therefore, every method which follows Bisection algorithm to divide Bézier curve, can generate shorter Bézier curve segments, that needs more number of circular arcs than minimal approximation solution.
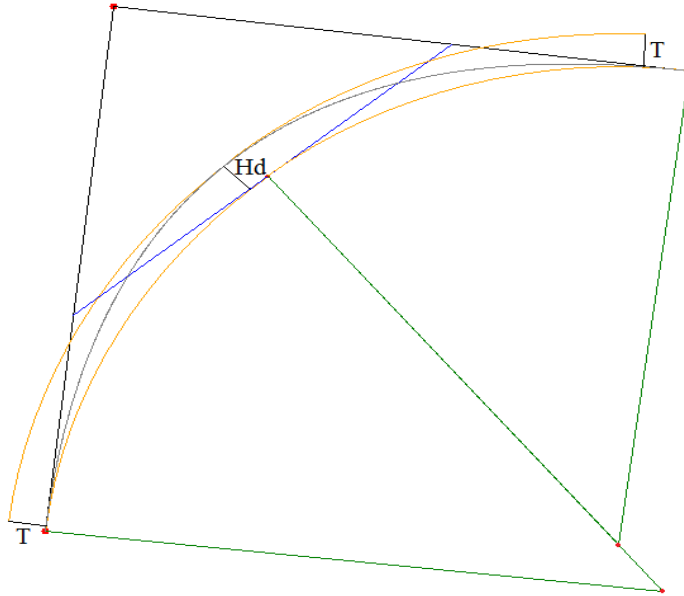


Figure 9: Hausdorff Distances and Tolerance Band relationship.
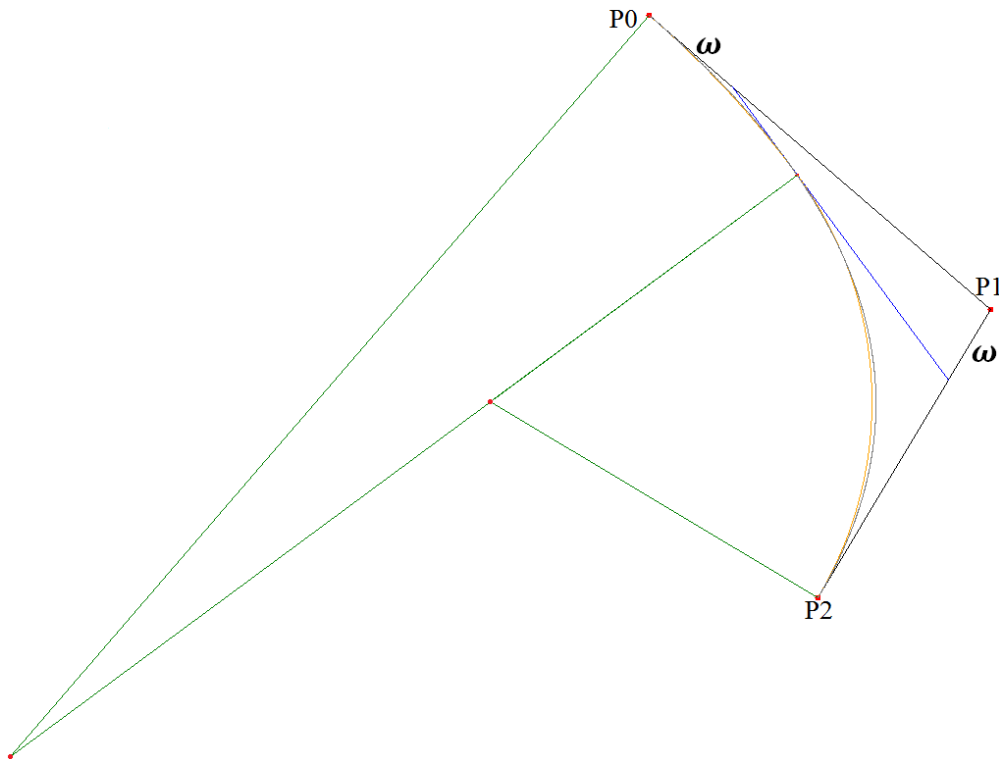
## 3. Quadratic Bézier-Like

In the previous section, I have explained the difficulties and complexities of quadratic Bézier curve approximating by Circular arcs.

One of those difficulties was to find Hausdorff distance and using a Tolerance band to approximating by Circular arcs.

We can define a kind of Bézier-Like curve to solve this problem. Bézier-Like curves are familiar with Bézier curve and using mostly the Bisection algorithm as a curve constructor. A class of Bézier-like curves is presented in [7].

In this section, a new kind of Bézier-Like curve has defined which is as closer as possible to Quadratic Bézier curve and to find Hausdorff distance and using a Tolerance band to approximating by Circular arcs is easier for this new type of curve.

Finding Hausdorff distance between quadratic Bézier curve and corresponding biarc would be quite easy when the control segments are the same size.

In this case, just one arc segment is tangent with quadratic Bézier curve at two end points and Hausdorff distance is the distance between the middle points of quadratic Bézier curve with center of the arc segment minus the arc segment radius. If the control segments are not the same size, then there is a tangent line on the quadratic Bézier curve, which can create two new quadratic Bézier curve with equal control segments. The new end point of these two new Bézier curve is not located on the previous Bézier curve and that create a new type of curve close to quadratic Bézier curve. Equation 10-12 and Figure 10 explain how to find the new control point.



Figure 10: Bézier coefficient to find biarc

Equation 10:
$$a = ||P_0 - P_1||, \ \ b = ||P_1 - P_2||, \ \ c = ||P_0 - P_2||, \ \ \cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$

Equation 11:
$$(\omega a + (1-\omega)b)^2 = ((1-\omega)a)^2 + (\omega b)^2 - 2(1-\omega)\omega ab \cos\theta, 0 \le \omega \le 1$$

Equation 12:
$$\omega = \frac{-(a+b)(b-3a)-c^2 \pm \sqrt{((a+b)(b-3a)+c^2)^2 - 4((a+b)^2-c^2)(a^2-b^2)}}{2((a+b)^2-c^2)}, 0 \le \omega \le 1$$

We can use the Bézier coefficient ω to construct the Bézier-Like curve like de-Casteljau algorithm and present a new algorithm as follow:

*Quadratic Bézier-Like* $(P_0^0, P_0^1, P_0^2)$
*{*
*If* $\left( \left\| P_0^0 - P_0^1 \right\| = \left\| P_0^1 - P_0^2 \right\| \right)$
   *{*

      $P_1^0 \leftarrow \frac{P_0^0 - P_0^1}{2} + P_0^1;$

      $P_1^1 \leftarrow \frac{P_0^1 - P_0^2}{2} + P_0^2;$

      $P_2^0 \leftarrow \frac{P_1^0 - P_1^1}{2} + P_1^1;$

      *Draw*$(P_2^0);$
      *Quadratic Bézier Like* $(P_0^0, P_1^0, P_2^0)$
      *Quadratic Bézier Like* $(P_2^0, P_1^1, P_0^2)$
   *}*
*Else*
   *{*

      $\omega \leftarrow$ *find the Bezier coefficient to divide* $(P_0^0, P_0^1, P_0^2);$
      $P_1^0 \leftarrow (P_0^0 - P_0^1)\omega + P_0^1;$
      $P_1^1 \leftarrow (P_0^1 - P_0^2)\omega + P_0^1;$
      $P_2^0 \leftarrow$ *find the touch point of biarc on tangent line* $(P_1^0, P_1^1);$
      *Draw*$(P_2^0)$
      *Quadratic Bézier Like* $(P_0^0, P_1^0, P_2^0)$
      *Quadratic Bézier Like* $(P_2^0, P_1^1, P_0^2)$
   *}*
*}*

This curve is constructed by Bisection algorithm but the calculation of Hausdorff Distance between this curve segments and corresponding biarc needs just *O(1)* processing time.

The quadratic Bézier-Like curve approximation by circular arc is much easier than Quadratic Bézier curve and it creates optimizer results. In additional we can calculate directly Hausdorff distance easier than quadratic Bézier curve to make an algorithm such as the last section.

Or we can use the Bézier coefficient ω easily in a algorithm like biarc method and present a new algorithm for quadratic Bézier-Like curve. We should just replace Draw($P_2^0$) with this pseudo code in the first part of the recursion function and remove Draw($P_2^0$) in the second part.

Figure 11 is shown how much the new curve is closer to Quadratic Bézier curve and Figure 12 is shown how easy this new curve approximate by circular arcs.

$$Ca(C, r) \leftarrow find\ Circular\ Arc\left(P_0^0, P_0^1, P_0^2\right);$$

$$If\left(\left|r - \left|\left|P_2^0 - C\right|\right|\right| < Tolerance\ band\right)$$

$$\{$$

$$\qquad Draw\ Circular\ Arc(Ca(C, r));$$
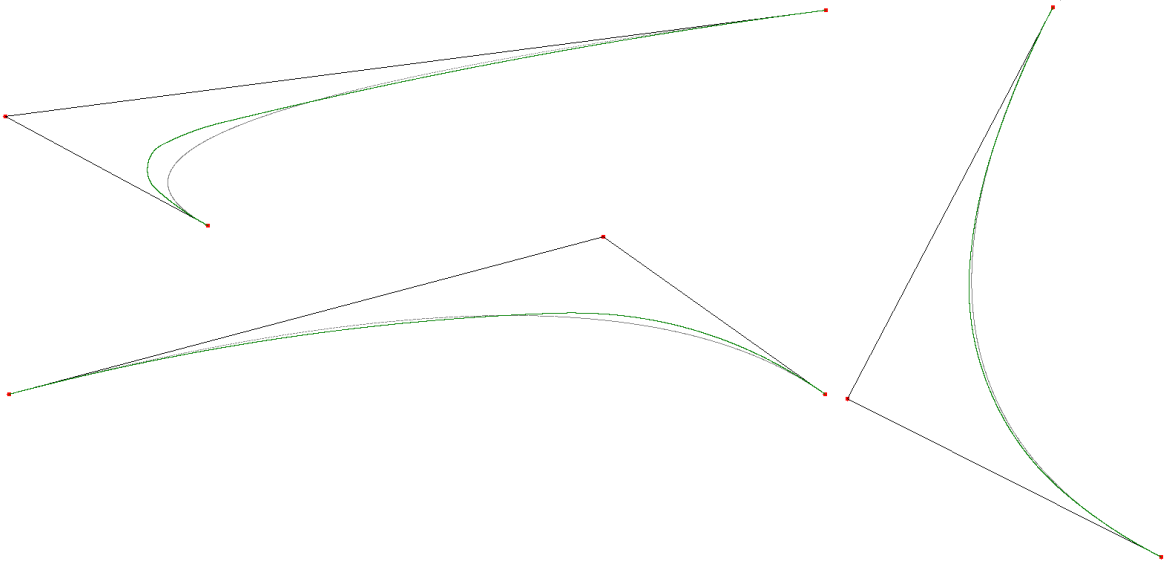
$$\qquad Return;$$

$$\}$$

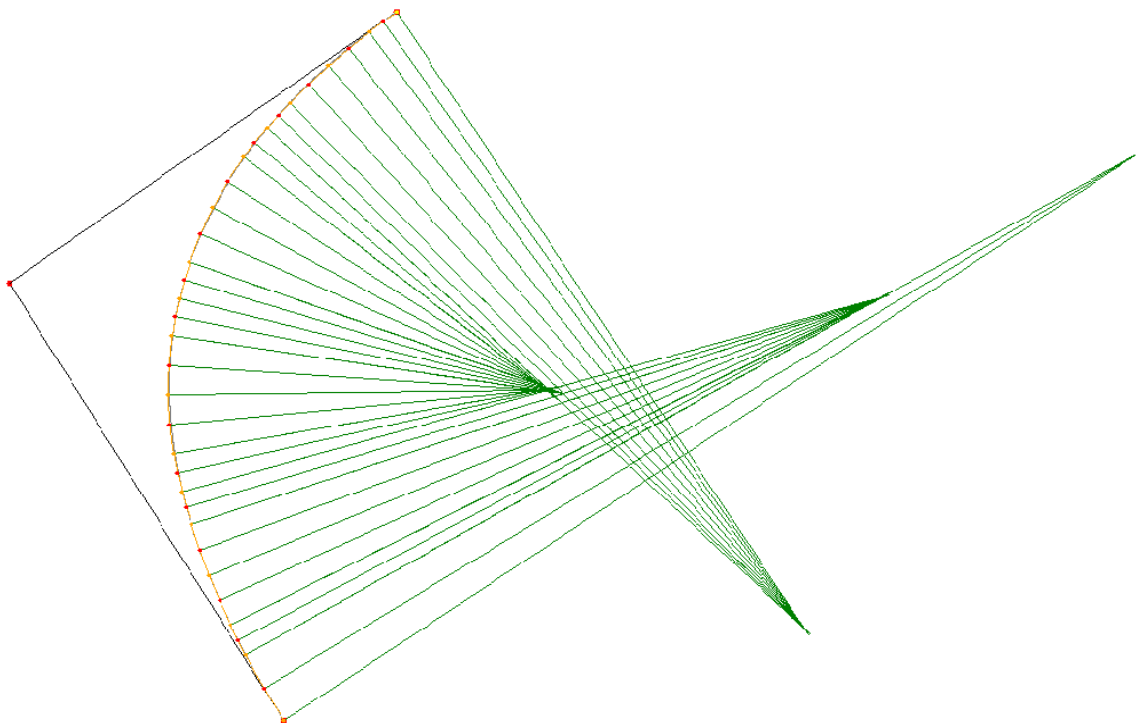Figure 11: Quadratic Bézier curve (gray) and quadratic Bézier-Like curve (green)

Figure 12: Some examples of quadratic Bézier-like curve approximation by circular arc (orange) and corresponding quadratic Bézier curve (gray)

## 4. Hausdorff Distance Equation

The Hausdorff distance is obtained from the maximum distance between $p(s_0)$ and $q(t_0)$ satisfying Equation 13, when they have the same end points, as shown in Figure 13. Thus to get the Hausdorff distance it requires to solve nonlinear system of two variables as in the Equation 13.

Equation 13: $\qquad p'(s_0) \parallel q'(t_0) \qquad$ and $\qquad p'(s_0) \perp p(s_0)q(t_0)$
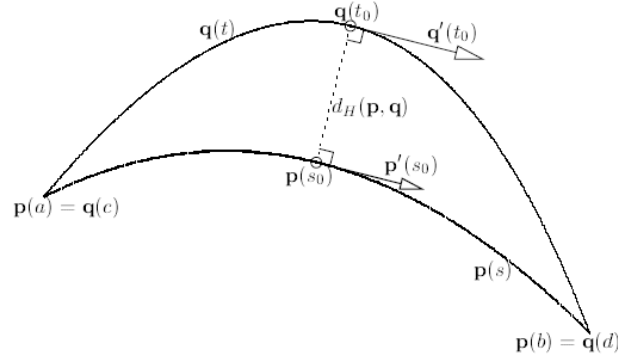


Figure 13: Hausdorff distance between two curves $p(s)$ and $q(t)$
$$s \in [a,b] \qquad t \in [c,d]$$

Ahn presented in [5], the exact Hausdorff distance between the offset curve of quadratic Bézier curve and its quadratic approximation by solving a nonlinear system of two variables such as in Equation 13.
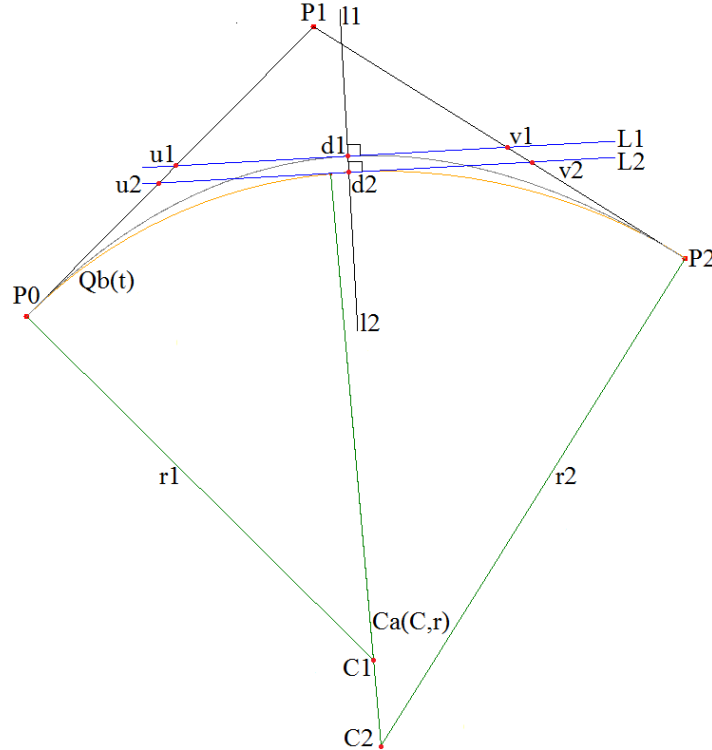


Figure 14: Hausdorff distance between the quadratic Bézier curve and the corresponding biarc approximation.

In this section, a new method is presenting by using a formula for the Hausdorff distance between the quadratic Bézier curve and the corresponding biarc approximation. It obtains the quadratic Bézier curve segment within a tolerance band by using this formula for the Hausdorff distance.

On based of Equation 13, we can calculate the Hausdorff distance between quadratic Bézier curve and the corresponding biarc approximation if two lines l1 and l2 are matched together as described in Figure 14.

Assume $L1$ is the tangent line on the quadratic Bézier curve $Qb(t)$ at $d1$ and $l1$ is the orthogonal line of $L1$ at $d1$. $L2$ is the tangent line of the corresponding biarc $Ca(C, r)$ at $d2$ that is parallel with $L1$ and $l2$ is the orthogonal line of $L2$ at $d2$ .

## 4.1.    The New Method

Every quadratic Bézier curve segment without the parabolic peak point has two Distance value which the maximum one is Hausdorff Distance as it is shown in Figure 15.
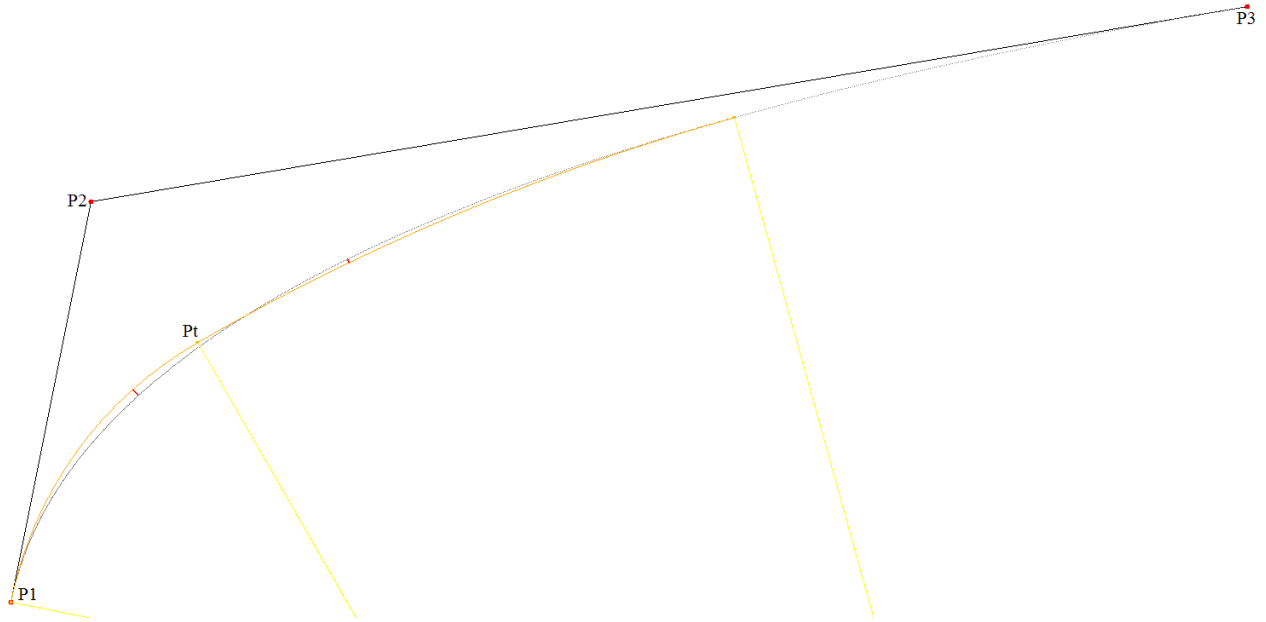


Figure 15: Two Distance values and Hausdorff Distance.

The new method is not using the Bisection Algorithm and it keeps Hausdorff Distance equal to tolerance band.

This method does not need a huge processing time to calculate Hausdorff Distance. If the quadratic Bézier curve contains the parabolic peak point then we have to divide it on this point and at the end we can merge two adjacent circular arcs on the parabolic peak point. The intersection point of each biarc to cover a curve segment touches a tangent line on the curve.

Algorithm of the new method works by 3 main steps as follow:
In the first step we have to draw the adjacent biarc belong to the shortest control line by the Hausdorff Distance as a tolerance band such as Figure 16.
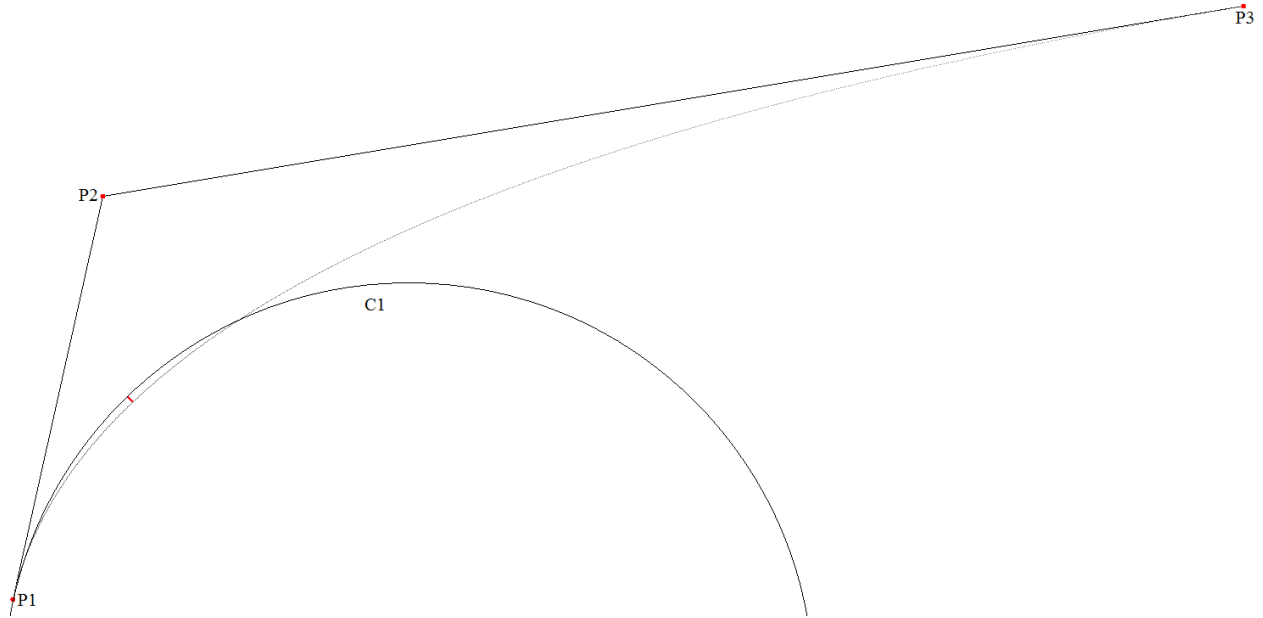


Figure 16: The first step of new method.

In the second step we are drawing *L1* that is the tangent line of the corresponding biarc *C1* at Pt and the quadratic Bézier curve at *Ps* such as Figure 17.
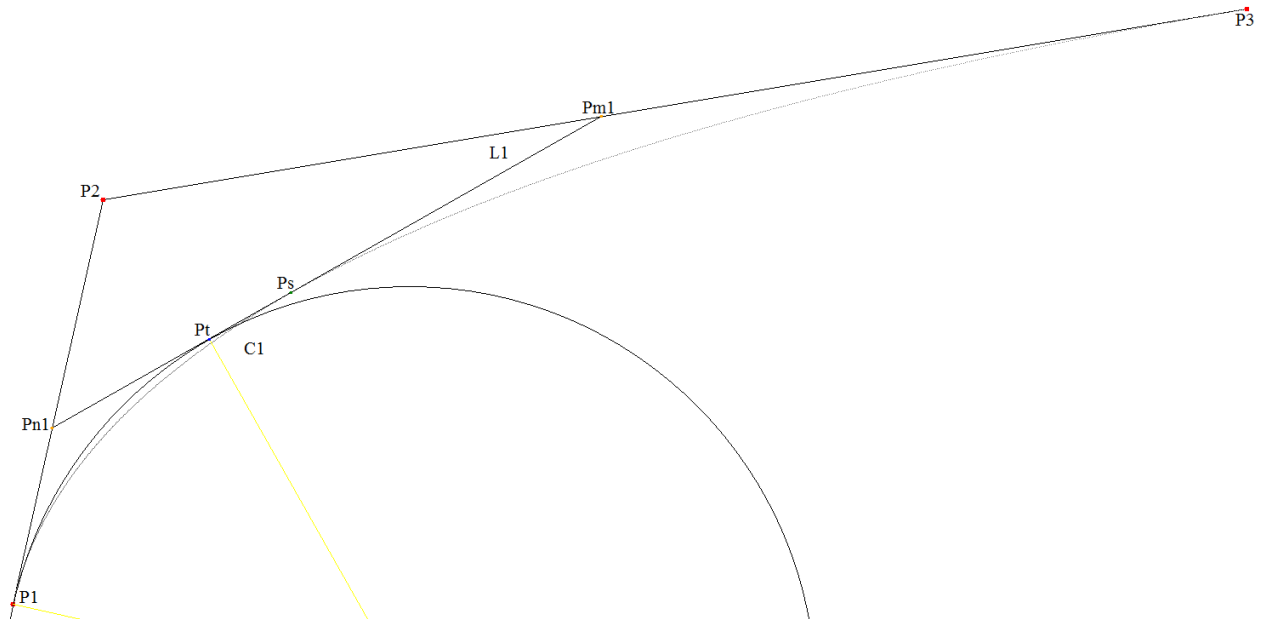


Figure 17: The second step of new method.

At the end we find the tangent biarc *C2* on the quadratic Bézier curve at *Pr* that is bigger than *C1* and touch it at *Pt* such as Figure 18.
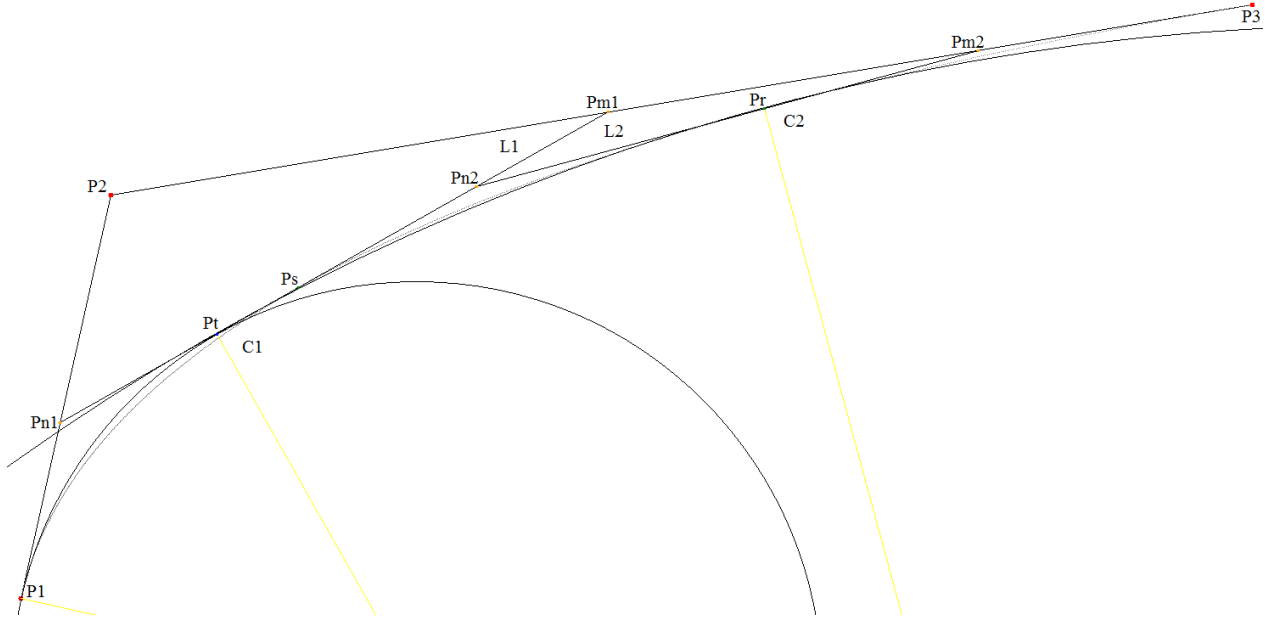
Figure 18: The third step of new method.

In this algorithm these 3 main steps should repeat on the rest of curve till end.

## 4.2.   Hausdorff Distance Optimization

There is a possibility to improve the new method by Hausdorff Distance optimization that means both of the biarc are belong to the Hausdorff Distance. Therefore they must have the same value to create the longest biarc to cover a curve segment as it is shown in Figure 19.
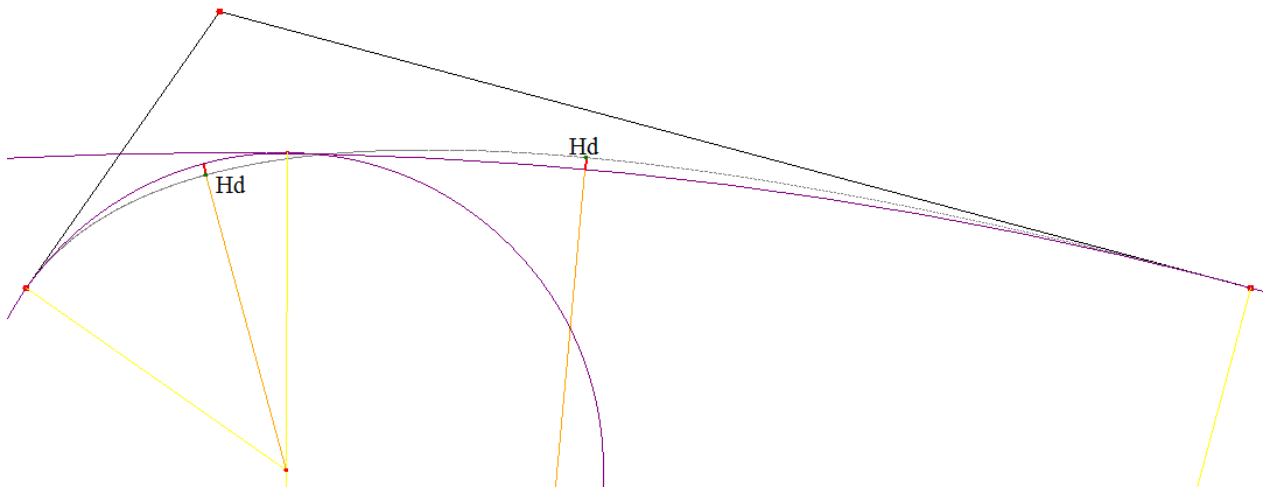
Figure 19: Two Distance values equal to Hausdorff Distance.

The Hausdorff Distance optimization creates less number of biarc. In the Figure 20 you can compare the result of the new method with and without Hausdorff Distance optimization. Figure 20.a has created by the new method without Hausdorff Distance optimization and it contains 21 arc segments and Figure 20.b used Hausdorff Distance optimization and creates 15 arc segments.
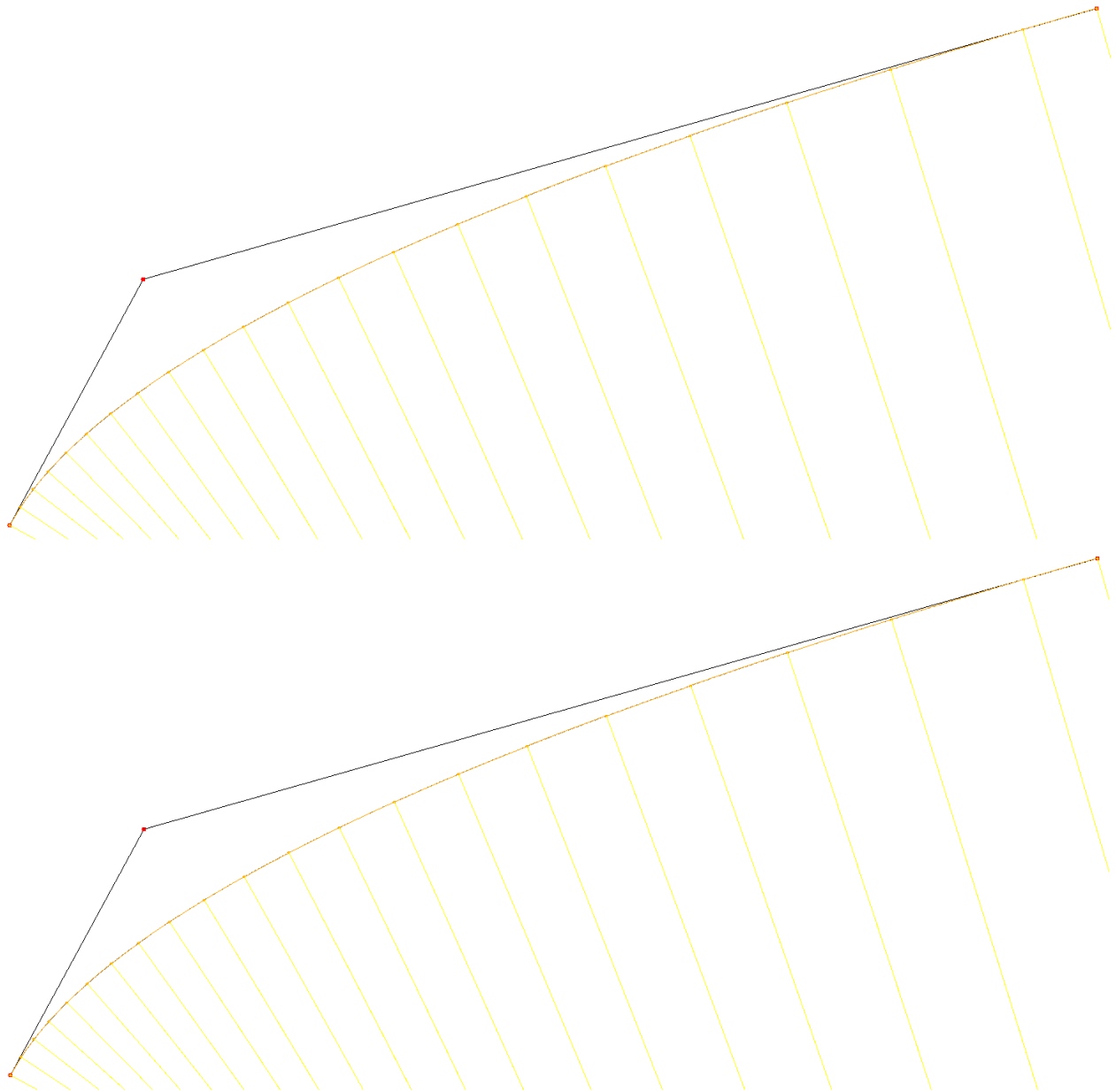


Figure 20:
a) For every curve segment just one of the biarc is related to Hausdorff Distance.
b) For every curve segment both of the biarc is related to Hausdorff Distance.

## 5. References

[1] D.J. Walton and D.S. Meek, Approximating of quadratic Bézier curves by arc Splines, Elsevier Journal, Computational and Applied Mathematics, No. 54, pp. 107-120, 1994.

[2] Young Joon Ahn, Hong Oh Kim and Kyoung Yong Lee, G1 arc spline approximation of quadratic Bézier curves, Elsevier Journal, Computer-Aided Design, Vol. 30, No. 8, pp. 615-620, 1998.

[3] J. H. Yong, S. M. Hu, J-G. Sun, Bisection algorithms for approximating quadratic Bézier curves by G1 arc splines, Elsevier Journal, Computer-Aided Design, No. 32, pp. 253-260, 2000.

[4] A. Riškus, Approximation of a cubic Bézier curve by circular arcs and vice versa, Journal of Information Technology and Control (ITC), Vol.35, No. 4, pp. 371-378, 2006.

[5] Young Joon Ahn, Hausdorff distance between the offset curve of quadratic Bézier curve and its quadratic approximation, Communications of the Korean Mathematical Society, Vol 22, No. 4, pp. 641-648, 2007.

[6] Hyungjun Park, Error-bounded biarc approximation of planar curves, Elsevier Journal, Computer-Aided Design, No. 36, pp. 1241-1251, 2004.

[7] Qinyu Chen, Guozhao Wang, A class of Bézier-like curves, Elsevier Journal, Computer-Aided Geometric Design, No. 20, pp. 29-39, 2003.

[8] D.J. Walton, D.S. Meek, Approximation of a planar cubic Bézier spiral by circular arcs, Elsevier Journal, Computational and Applied Mathematics, Vol 75, No.1, pp. 47-56, 1996.