Composition of polynomials, lacunarity, and linear recurrences

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Polynomial decomposition questions have been the subject of various number-theoretical investigations, starting with Ritt in the 1920's. He showed that in any representation of a given complex polynomial f of the shape $f = f_1 \circ f_2 \circ \cdots \circ f_n$, where the f_i, $i = 1, \ldots, n$, cannot be further decomposed into lower-degree polynomials, n is determined uniquely as is the multiset of degrees of the f_i. In the meantime, many structural and algorithmic results in this area have been achieved. In this thesis, decomposition questions are investigated, first in combination with lacunary polynomials and, in the second part, for linear recurrence sequences of complex polynomials. It was shown by Zannier that if $f = g \circ h$ is lacunary, i.e. it has only a given number of terms (that is assumed to be small compared to its degree), then g and h have this property as well, provided $h(x) \neq ax^n + b$. Following his ideas, explicit upper bounds for the number of terms of $h(x)$, written as a rational function, are given. Moreover, some structural results for decomposable polynomials which are elements of a linear recurrence sequence are presented. For binary recurrence sequences $(G_n)_{n\geq 0}$ in $\mathbb{C}[x]$, it is shown that, if $G_n = g \circ h$ for some $g, h \in \mathbb{C}[x]$, where h is indecomposable, then, under certain assumptions and if a technical condition holds, either $\deg g$ is bounded by an explicitly computable constant, or $h(x)$ is of very special shape. The last part contains a complete algebraic description of all m-decompositions in a d-th order linear recurrence sequence having polynomial characteristic roots, one of which is strictly greater than the others. It is shown that all such decompositions can be described in finite terms, coming from a generic one, parametrized by an algebraic variety. In the proofs, concepts from the theory of algebraic function fields are applied. In particular, results connected to S-unit equations and the use of height functions, such as Brownawell and Masser's inequality, serve as a main tool.