hp-adaptive finite elements and the Finite Cell method

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The Finite Cell Method (FCM) combines a fictitious domain approach with a (higher order) finite element method. Its basic idea consists in replacing the possibly complicated physical domain by an enclosing domain of simple shape, which can easily be meshed. The variational formulation of the problem and its finite element discretization are defined on the enclosing domain and the geometry of the physical domain is incorporated via an indicator function, which necessitates the application of appropriate quadrature schemes.

In this thesis, we construct C^k basis functions on meshes with hanging nodes and develop quadrature techniques and error control for the FCM.

In [1], we present an easy treatment of (multi-level) hanging nodes in hp-finite elements for 2D meshes. Its simplicity is due to the fact that the connectivity matrices can be directly computed in a row-wise fashion. This is achieved by a piecewise definition of each basis function on an appropriate union of elements. Numerical experiments compare the new approach to the conventional one with respect to accuracy and the conditioning of the resulting linear system. The new approach achieves nearly the same properties as the conventional one in the examples and provides an excellent trade-off between implementational complexity and solvability.

In [2], we generalize the methods from [1] to construct C^k basis functions for paraxial d-dimensional rectangular meshes typically used in the FCM. The technique allows for arbitrary hanging nodes and varying polynomial degrees. The use of shape functions based on Hermite and Gegenbauer polynomials enables the support of the basis functions to be independent of k. An appropriate indexing is introduced in order to prove the differentiability. The construction is also suitable for C^0 finite elements on meshes with generalized rectangles. Numerical examples illustrate the feasibility of the approach. Finally, some aspects concerning the condition number of the stiffness matrix are discussed.

In [3], we present two techniques to obtain quadrature rules suitable for the FCM. The first technique is based on a combination of the moment-fitting method with an optimization strategy in order to reduce the number of the quadrature points. The second technique applies the smart octree, which generates an integration mesh consisting of generalized rectangles suited to the domain. Several numerical examples demonstrate the efficiency of the resulting quadrature rules.

In [4], we derive a reliable residual-based error estimator for the FCM suitable in the context of hp-adaptivity. The proof relies on standard arguments of residual-based a posteriori error control, but includes several modifications to cope with the complex geometry of the cut elements. The efficiency of the error estimator is discussed by means of an artificial example that yields an efficiency index depending on the mesh-family parameter h. However, numerical experiments for h- and p-uniform, hp-geometric, and h-adaptive refinements suggest global efficiency with a large overestimation on only few cut elements.

In [5], we establish a goal-oriented error control for the FCM based on the dual weighted residual method. The error identity allows for a combined treatment of the discretization and the quadrature error. We present an adaptive strategy with the aim to balance the two error contributions and demonstrate its performance for several numerical examples in 2D.

References

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