Regularity Theory and Gradient Flows of Geometric Curvature Energies for Curves

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Geometric curvature energies have proved to be efficacious in measuring the geometry of embedded curves and surfaces. The quantities are designed to model self-avoidance phenomena and exhibit desirable regularizing properties. Of particular interest is the investigation of such energy quantities in the context of geometric knot theory, where one seeks optimal representatives as local minimizers within every knot class. The O'Hara knot energies, reminiscent of the electrostatic energy of an electric charge, are the first energies suggested for that purpose. Other prominent knot energies make comparisons between the circumcircle and the osculating circle of a curve; the averages of reciprocal powers of the respective radii produce potential energies such as the integral Menger curvature or the tangent-point energies.

In the first part of this thesis, we investigate the implications of finite knot energy for knotted curves. Finite energy generally has a regularizing impact on curves and especially on stationary points of the energy as they are constructed to correspond with optimal knot configurations. We show for sub-critical non-degenerate O'Hara knot energies and generalized integral Menger curvature energies that smooth critical points of finite energy are also analytic. The main tools are Cauchy's method of majorants, a fractional Leibniz rule, and an iteration argument. Subsequently, we study stationary points of finite generalized tangent-point energies in the scale-invariant case. We develop a theory of locally critical embeddings and prove Hölder regularity of the unit tangent. We apply various techniques for that, amongst others we build a bridge from the considered knot energies to harmonic analysis.

The second part of the thesis focuses on evolution equations associated with geometric curvature energies. The aim is to solve the negative L2-gradient flow of the p-elastic energy with $p > 2$ for curves and thereby lay a foundation for future research on gradient flows of degenerate knot energies. By means of approximate normal graphs and minimizing movements, we establish short-time existence and give a lower bound on the solution's lifetime. We improve the result by a different ansatz, the vanishing viscosity method. Employing this approach, we first determine short-time existence, long-time existence, and asymptotics for the gradient flow of regularized energies. Finally, we transfer our findings to the actual equations.